

INTERSECTION THEORY IN ALGEBRAIC COBORDISM

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ABSTRACT. We give a more detailed construction of the operation “intersection with a pseudo-divisor” in algebraic cobordism. Using arguments in [6, §6.2, 6.3], this gives a new proof of the contra variant functoriality of algebraic cobordism for l.c.i. morphisms of schemes of finite type over a field of characteristic zero.

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INTRODUCTION

One important aspect of algebraic cobordism, as constructed in [6], is the existence of a good theory of pull-back maps for arbitrary l.c.i. morphisms (always working over a field of characteristic zero). The key ingredient which makes this possible is the *intersection map*

$$D(-) : \Omega_*(X) \rightarrow \Omega_{*-1}(|D|)$$

for an effective Cartier divisor (more generally a *pseudo-divisor*) D with support $|D|$ on a finite type k -scheme X . The construction of the intersection map in [6, §6.2, 6.3] is accomplished with the help of a refined cobordism group $\Omega_*(X)_D$, which admits an explicitly defined intersection map

$$D(-)_D : \Omega_*(X)_D \rightarrow \Omega_{*-1}(|D|)$$

and a “forgetful map” $\Omega_*(X)_D \rightarrow \Omega_*(X)$. The forgetful map is shown to be an isomorphism [6, theorem 6.4.12]; the map $D(-)$ is defined by

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composing $D(-)_D$ with the inverse of the forgetful map. Auxiliary refinements $\Omega_*(X)_{D|D'}$ of $\Omega_*(X)$ are also defined, and are used to prove properties of the intersection map, most importantly, the commutativity property $D(D'(x)) = D'(D(x))$ in $\Omega_{*-2}(|D| \cap |D'|)$ for $x \in \Omega_*(X)$ and pseudo-divisors D, D' on X .

This approach has been criticized, as it is not proven (nor is it necessary for the construction) that the auxiliary groups $\Omega_*(X)_{D|D'}$ are themselves isomorphic to $\Omega_*(X)$. Fulton has suggested that one should be able to define a series of refined groups $\Omega_*(X)_{D_1, \dots, D_n}$ for pseudo-divisors D_1, \dots, D_n on X , related by intersection maps and forgetful maps, so that all forgetful maps are isomorphisms, and giving rise to all necessary relations among the intersection maps.

Carrying out such a program is the purpose of this paper. We define groups $\Omega_*(X)_{D_1, \dots, D_n}$, with intersection maps

$$D(-)_{D_*} : \Omega_*(X)_{D, D_1, \dots, D_n} \rightarrow \Omega_{*-1}(|D|)_{D_1, \dots, D_n}$$

and forgetful maps

$$\text{res}_{D_*, D/D_*} : \Omega_*(X)_{D_1, \dots, D_n, D} \rightarrow \Omega_*(X)_{D_1, \dots, D_n}.$$

We show that the forgetful maps are isomorphisms and that the resulting intersection maps

$$D(-) : \Omega_*(X) \rightarrow \Omega_{*-1}(|D|)$$

have all the properties needed to give rise to pullback maps in algebraic cobordism for l.c.i. morphisms via the technique of deformation to the normal bundle. The groups $\Omega_*(X)_D$ (for a single pseudo-divisor) and the intersection map $D(-)_D : \Omega_*(X)_D \rightarrow \Omega_{*-1}(|D|)$ are exactly the same as those defined in [6]. The passage from the intersection maps to the l.c.i. pullbacks is not discussed here; the arguments and constructions of [6, §6.5, §6.6], relying on the properties of the intersection map detailed in proposition 6.3, will do that.

The requirement that each forgetful map should be an isomorphism has led to a definition of the groups $\Omega_*(X)_{D_1, \dots, D_n}$ that for $n = 2$ differs from the groups $\Omega_*(X)_{D_1|D_2}$ defined in [6]. We hope that the more uniform approach pursued here has made the arguments more transparent and natural.

We fix a base-field k , let \mathbf{Sch}_k denote the category of separated finite type k -schemes and let \mathbf{Sm}_k denote the full subcategory of smooth quasi-projective k -schemes. We often drop the qualifier “separated” and refer to an object of \mathbf{Sch}_k as a finite type k -scheme. Although many of the constructions do not require resolution of singularities, or characteristic zero, we will assume that k has characteristic zero. It would be possible to work over a perfect field of positive characteristic, assuming resolution of singularities, but we have preferred to avoid this (at present illusory) generality as it allows us to shorten the argument for lemma 1.16 by using Bertini’s theorem. The corresponding result [6, lemma 6.1.13] was proven without the use of Bertini’s theorem and a similar proof would work in this setting.

We will be using only the following properties of “resolution of singularities over k ”: The field k should be perfect. For each integral $X \in \mathbf{Sch}_k$, there is a projective birational map $q : Y \rightarrow X$ with Y in \mathbf{Sm}_k such that q is an isomorphism over the smooth locus in X . In addition, if D is an effective Cartier divisor on X , whose restriction to a smooth open subscheme $U \subset X$ is a simple normal crossing divisor, there is a projective birational map $q : Y \rightarrow X$ with Y in \mathbf{Sm}_k such that q is an isomorphism over U , and $q^*(D) + E$ is a simple normal crossing divisor on Y , where E is the exceptional locus of q . For proofs of these facts over a field of characteristic zero, we refer the reader to [1, 2, 3, 5].

In section 1 we define the refined cobordism groups $\Omega_*(X)_{D_1, \dots, D_n}$. We introduce the notion of admissibility of an effective simple normal crossing divisor E with respect to $D_* := D_1, \dots, D_n$ and define the divisor class $[E \rightarrow |E|]_{D_*} \in \Omega_*(|E|)_{D_*}$ for an admissible simple normal crossing divisor E ; these classes will play a central role in our definition of the intersection map. The groups $\Omega_*(X)_{D_*}$ admit 1st Chern class operators for line bundles on X and we discuss some basic properties of these operators and their relation with the divisor classes. In section 2 we construct the intersection map on generators for $\Omega_*(X)_{D_*}$ and derive some of its important properties. In section 3, we show that intersection map descends to a map on $\Omega_*(X)_{D_*}$. We establish the relation of commutativity of intersection maps in section 4 and show under certain technical conditions the equality of intersection with linearly equivalent divisors. We use these relations and a construction of explicit “distinguished liftings” to prove that the forgetful maps are isomorphisms, theorem 5.11, in section 5. We conclude in section 6 with the definition of the intersection map on $\Omega_*(-)$ and a proof of its main properties.

We wish to mention explicitly, that although we refer to our theorem 5.11, stating that the forgetful maps are isomorphisms, as a ‘moving lemma’, the arguments rely for their algebro-geometric input on resolution of singularities. Techniques using projecting cones or any of the other aspects of the classical Chow’s moving lemma or its more modern extension to moving lemmas for Bloch’s cycle complexes do not appear. If one would return to the Chow groups via the isomorphism $\Omega^* \otimes_{\mathbb{L}} \mathbb{Z} \cong \mathrm{CH}^*$ of [6, Theorem 4.5.1], one recovers Fulton’s definition of intersection with a pseudo-divisor, and our results say nothing about the earlier proofs of the contravariant functoriality of the Chow groups of smooth varieties that rely on Chow’s moving lemma. The added difficulty here is due to the fact that the generators of algebraic cobordism are smooth varieties rather than integral closed subschemes and one needs a modified version of intersection with a pseudo-divisor that takes this into account.

1. REFINED COBORDISM

1.1. Pseudo-divisors. Let X be a finite type k -scheme. Following Fulton [4], a *pseudo-divisor* D on X is a triple $D := (Z, \mathcal{L}, s)$, where $Z \subset X$ is a closed subset, \mathcal{L} is an invertible sheaf on X , and s is a section of \mathcal{L} on X , such that the subscheme $s = 0$ has support contained in Z ; one identifies triples (Z, \mathcal{L}, s) , (Z, \mathcal{L}', s') if there is an isomorphism $\phi : \mathcal{L} \rightarrow \mathcal{L}'$ with $s' = \phi(s)$. In particular, having fixed \mathcal{L} , the section s is determined exactly up to a global unit on X . If we have a morphism $f : Y \rightarrow X$, we define $f^*(Z, \mathcal{L}, s) := (f^{-1}(Z), f^*\mathcal{L}, f^*s)$; clearly $(fg)^*(D) = g^*(f^*D)$ for a pseudo-divisor D . Also, an effective Cartier divisor D on X uniquely determines a pseudo-divisor $(|D|, \mathcal{O}_X(D), s_D)$, where $s_D : \mathcal{O}_X \rightarrow \mathcal{O}_X(D)$ is the canonical section and $|D|$ is the support of D .

We call Z the *support* of a pseudo-divisor $D := (Z, \mathcal{L}, s)$, and write $Z = |D|$. We let $\text{div} D$ denote the subscheme $s = 0$, and write $\mathcal{O}_X(D)$ for \mathcal{L} . If X is in \mathbf{Sm}_k , if $|D| = |\text{div} D|$ and if this subset has pure codimension one on X , then we identify D with the Cartier divisor $\text{div} D$.

We will not be needing the full flexibility of all pseudo-divisors. In this paper, we will always assume that the support Z of a pseudo-divisor $D := (Z, \mathcal{L}, s)$ is given by the closed subset $s = 0$. Thus, a pseudo-divisor on a smooth irreducible Y is either $(|D|, \mathcal{O}_Y(D), s_D)$ for an effective Cartier divisor D on Y , or $(Y, \mathcal{L}, 0)$ for some invertible sheaf \mathcal{L} on Y .

The zero pseudo-divisor is $(\emptyset, \mathcal{O}_X, 1)$. If we have pseudo-divisors $D = (Z, \mathcal{L}, s)$ and $D' = (Z', \mathcal{L}', s')$, define $D + D' = (Z \cup Z', \mathcal{L} \otimes \mathcal{L}', s \otimes s')$. A pseudo-divisor C is a *sub-pseudo-divisor* of a pseudo-divisor D if there is an effective Cartier divisor E with $C + E = D$. We say that C is *supported in* D if C is a sub-pseudo-divisor of mD for some integer $m \geq 1$. If $f : Y \rightarrow X$ is a morphism of finite type k -schemes and C and D are pseudo-divisors on X with C supported in D , then $|f^*C| \subset |f^*D|$ and, if f^*D is a Cartier divisor on Y , necessarily effective, then f^*C is also an effective Cartier divisor on Y , with support contained in $|f^*D| = |\text{div} f^*D|$.

1.2. Refined cobordism cycles. Let X be a finite type k -scheme and D_1, \dots, D_r pseudo-divisors on X . We proceed to define a series of groups

$$\mathcal{Z}_*(X)_{D_*} \rightarrow \underline{\mathcal{Z}}_*(X)_{D_*} \rightarrow \underline{\Omega}_*(X)_{D_*} \rightarrow \mathbb{L}_* \otimes \underline{\Omega}_*(X)_{D_*} \rightarrow \Omega_*(X)_{D_*}$$

analogous to the sequence

$$\mathcal{Z}_*(X) \rightarrow \underline{\mathcal{Z}}_*(X) \rightarrow \underline{\Omega}_*(X) \rightarrow \mathbb{L}_* \otimes \underline{\Omega}_*(X) \rightarrow \Omega_*(X)$$

used to define $\Omega_*(X)$ in [6, §2.4]; in this section we construct the group $\mathcal{Z}_*(X)_{D_*}$.

Let $E = \sum_{i=1}^m n_i E_i$ be an effective simple normal crossing on a scheme $W \in \mathbf{Sm}_k$ with irreducible components E_1, \dots, E_m . For each $J \in \{0, 1\}^m$, $J = (j_1, \dots, j_m)$, we have the *face*

$$E^J := \cap_{j_i=1} E_i,$$

which is smooth over k and has codimension $|J| := \sum_i j_i$ on W . All Cartier divisors we will be using here will be effective and will often refer to an effective simple normal crossing divisor as a simple normal crossing divisor.

Let X be a finite type k -scheme. We recall from [6, §2.1.2] the set $\mathcal{M}(X)$ of isomorphism classes of projective morphisms $f : Y \rightarrow X$, with Y in \mathbf{Sm}_k (where “isomorphism” means isomorphism over X). $\mathcal{M}(X)$ is a monoid under disjoint union and is the free monoid on the isomorphism classes of $f : Y \rightarrow X$ with Y irreducible.

Let $\mathcal{M}(X)_D$ be the submonoid of $\mathcal{M}(X)$ generated by $f : Y \rightarrow X$, with Y irreducible, and with either $f(Y) \subset |D|$, or with $\text{div } f^*D$ a simple normal crossing divisor on Y . We extend this construction as follows.

Definition 1.1. Let X be a finite type k -scheme, and let D_1, \dots, D_r be pseudo-divisors on X .

i) Let $f : Y \rightarrow X$ be a morphism, with Y irreducible. Let $I_f = \{i \mid |f^*D_i| \neq Y\}$. If $I_f \neq \emptyset$, let i_0 be the smallest $i \in I_f$ and call D_{i_0} the *leading* pseudo-divisor for f . If $I_f = \emptyset$, we say that f has no leading pseudo-divisor.

ii) Let $f : Y \rightarrow X$ be a morphism with $Y \in \mathbf{Sm}_k$. Suppose Y is irreducible. We say f that is *admissible* with respect to D_1, \dots, D_r if for each s , $1 \leq s \leq r$, the scheme-theoretic intersection $\cap_{i=1}^s \text{div } f^*D_i$ is either Y , or is empty, or is a Cartier divisor with support a simple normal crossing divisor on Y . If Y is not irreducible, f is admissible if the restriction of f to each irreducible component of Y is so. The submonoid $\mathcal{M}(X)_{D_1, \dots, D_r}$ of $\mathcal{M}(X)$ is the subset consisting of those $f : Y \rightarrow X$ that are admissible with respect to D_1, \dots, D_r .

iii) Let $f : Y \rightarrow X$ be a morphism, with $Y \in \mathbf{Sm}_k$. An effective Cartier divisor E on Y is called *admissible* with respect to D_1, \dots, D_r if E is a simple normal crossing divisor on Y and, for each face E^J of E , the composition $E^J \rightarrow Y \rightarrow X$ is admissible with respect to D_1, \dots, D_r .

When the collection of pseudo-divisors is understood we sometimes write $\mathcal{M}(X)_{D_*}$ for $\mathcal{M}(X)_{D_1, \dots, D_r}$. If D_1, \dots, D_r and D'_1, \dots, D'_s are two sequences of pseudo-divisors on X , we write D_*, D'_* for the sequence $D_1, \dots, D_r, D'_1, \dots, D'_s$. For an X -scheme $f : X' \rightarrow X$, we write $f^*(D_*)$ for the sequence of pseudo-divisors $f^*(D_1), \dots, f^*(D_r)$ and often write $\mathcal{M}(X')_{D_*}$ for $\mathcal{M}(X')_{f^*(D_*)}$ if the map f is understood.

Remarks 1.2. (1) For $f : Y \rightarrow X$ in $\mathcal{M}(X)$, f is in $\mathcal{M}(X)_{D_*}$ exactly when Id_Y is in $\mathcal{M}(Y)_{D_*}$.

(2) Suppose X is irreducible, D_1, \dots, D_r pseudo-divisors on X . Write $I_{\text{Id}_X} = \{i_0, i_1, \dots, i_s\}$ and let $D_j^{\text{ess}} = D_{i_j}$. Then $\mathcal{M}(X)_{D_*} = \mathcal{M}(X)_{D_*^{\text{ess}}}$. Also, $\mathcal{M}(X)_{\emptyset} = \mathcal{M}(X)$.

(3) Let $f : Y \rightarrow X$ be in $\mathcal{M}(X)_{D_*}$ with Y irreducible. Suppose that f has leading pseudo-divisor D_{i_0} and take s with $i_0 \leq s \leq r$. Let $D_{i_0}^1, \dots, D_{i_0}^m$ be the irreducible components of the simple normal crossing divisor $\text{div } f^*D_{i_0}$.

Then the intersection $E_s := \cap_{i=i_0}^s \operatorname{div} f^* D_i$ is an effective Cartier divisor of the form $\sum_j m_j D_{i_0}^j$; in particular, E_s is a simple normal crossing divisor on Y with $0 \leq E_s \leq \operatorname{div} f^* D_{i_0}$.

(4) The condition in definition 1.1(ii) is equivalent to the following: For $f : Y \rightarrow X$ with Y irreducible in \mathbf{Sm}_k , f is admissible if f has no leading pseudo-divisor or f has leading pseudo-divisor D_{i_0} and

- a) $\operatorname{div} f^* D_{i_0}$ is a simple normal crossing divisor on Y .
- b) For all s with $i_0 \leq s \leq r$, the scheme-theoretic intersection $\cap_{i=i_0}^s \operatorname{div} f^* D_i$ is a Cartier divisor on Y , or is empty.

Lemma 1.3. *Let D_1, \dots, D_r be pseudo-divisors on $X \in \mathbf{Sch}_k$, let $f : Y \rightarrow X$ be a morphism with $Y \in \mathbf{Sm}_k$, Y irreducible, and with $\operatorname{Id}_Y \in \mathcal{M}(Y)_{D_*}$. Let E be an effective Cartier divisor on Y . Suppose that D_{i_0} is the leading pseudo-divisor for f and that $E + \operatorname{div} f^* D_{i_0}$ is a simple normal crossing divisor on Y . Then E is admissible with respect to D_1, \dots, D_r .*

Proof. To simplify the notation, we may replace X with Y , D_* with $f^*(D_*)$ and f with Id_Y . We may therefore assume that X is in \mathbf{Sm}_k , is irreducible and $f = \operatorname{Id}_X$.

Under our assumption, E is a simple normal crossing divisor; so let F be an irreducible component of a face E^J of E and let $g : F \rightarrow X$ be the inclusion $F \rightarrow X$.

Suppose that F is not contained in $|D_{i_0}|$. Then D_{i_0} is also the leading pseudo-divisor for g . Since F is a component of a face of the simple normal crossing divisor $E + \operatorname{div} D_{i_0}$, $\operatorname{div} g^* D_{i_0}$ is a simple normal crossing divisor on F . Similarly, for all s , $i_0 \leq s \leq r$ the scheme-theoretic intersection on F , $\cap_{i=i_0}^s \operatorname{div} g^* D_i$ is just $g^{-1}(\cap_{i=i_0}^s \operatorname{div} D_i)$. As $\cap_{i=i_0}^s \operatorname{div} D_i$ is a Cartier divisor on X , with support in $\operatorname{div} D_{i_0}$, it follows that $\cap_{i=i_0}^s \operatorname{div} g^* D_i$ is a Cartier divisor on F .

Suppose that F is contained in $|D_{i_0}|$. If g has no leading pseudo-divisor, then clearly $F \rightarrow X$ is in $\mathcal{M}(X)_{D_*}$; if g has leading pseudo-divisor D_{i_1} , then $i_0 < i_1$ and $F \subset \operatorname{div} D_i$ for $i_0 \leq i < i_1$. Thus, $\operatorname{div} g^* D_{i_1} = g^{-1}(\cap_{i=i_0}^{i_1} \operatorname{div} D_i)$. Since $\cap_{i=i_0}^{i_1} \operatorname{div} D_i$ is a Cartier divisor on X with $0 \leq \cap_{i=i_0}^{i_1} \operatorname{div} D_i \leq \operatorname{div} D_{i_0} + E$, and F is a face of the simple normal crossing divisor $\operatorname{div} D_{i_0} + E$, it follows that $\operatorname{div} g^* D_{i_1}$ is a simple normal crossing divisor on F . Similarly, for all s with $i_1 \leq s \leq r$, $\cap_{i=i_1}^s g^* \operatorname{div} D_i$ is a Cartier divisor on F , and thus $g : F \rightarrow X$ is in $\mathcal{M}(X)_{D_*}$. \square

Lemma 1.4. *Let D_1, \dots, D_r be pseudo-divisors on $X \in \mathbf{Sch}_k$, let $f : Y \rightarrow X$ be a morphism with $Y \in \mathbf{Sm}_k$, Y irreducible, and with $\operatorname{Id}_Y \in \mathcal{M}(Y)_{D_*}$. Suppose that D_{i_0} is the leading pseudo-divisor for f . Then $\operatorname{div} f^* D_{i_0}$ is simple normal crossing divisor on Y , admissible with respect to D_{i_0+1}, \dots, D_r .*

Proof. As above, we reduce to the case $Y = X$, $f = \operatorname{Id}_Y$. Since Id_Y is in $\mathcal{M}(Y)_{D_*}$, it follows that $\operatorname{div} D_{i_0}$ is a simple normal crossing divisor on Y . From lemma 1.3 (with $E = \operatorname{div} D_{i_0}$), it follows that $\operatorname{div} D_{i_0}$ is admissible

with respect to D_1, \dots, D_r . If now F is a face of $\text{div} D_{i_0}$, then the inclusion $i_F : F \rightarrow Y$ is in $\mathcal{M}(Y)_{D_*}$. However, $Y = |D_i|$ for $i = 1, \dots, i_0 - 1$ and $F \subset |D_{i_0}|$, so either i_F has no leading pseudo-divisor or i_F has leading pseudo-divisor D_{i_1} with $i_0 < i_1 \leq r$. Therefore i_F is in $\mathcal{M}(Y)_{D_{i_0+1}, \dots, D_r}$; as F was an arbitrary face of $\text{div} D_{i_0}$, it follows that $\text{div} D_{i_0}$ is admissible with respect to D_{i_0+1}, \dots, D_r . \square

Recall from [6, definition 2.1.6] the notion of a cobordism cycle over X , namely, a tuple $(f : Y \rightarrow X, L_1, \dots, L_m)$ with $Y \in \mathbf{Sm}_k$, Y irreducible, f a morphism representing a class in $\mathcal{M}(X)$ and the L_i line bundles on Y . Two cobordism cycles $(f : Y \rightarrow X, L_1, \dots, L_m)$ and $(f' : Y' \rightarrow X, L'_1, \dots, L'_m)$ are isomorphic if there is an isomorphism $\phi : Y \rightarrow Y'$ over X and a permutation σ such that $\phi^* L'_i \cong L_{\sigma(i)}$ for $i = 1, \dots, m$. We have the free abelian group $\mathcal{Z}_*(X)$ on the isomorphism classes of cobordism cycles, graded by giving $(f : Y \rightarrow X, L_1, \dots, L_m)$ degree $\dim_k Y - m$.

Definition 1.5. For $X \in \mathbf{Sch}_k$ with pseudo-divisors D_1, \dots, D_r , we let $\mathcal{Z}_*(X)_{D_*}$ be the graded subgroup of $\mathcal{Z}_*(X)$ generated by the cobordism cycles $(f : Y \rightarrow X, L_1, \dots, L_m)$ with $f : Y \rightarrow X$ in $\mathcal{M}(X)_{D_*}$.

For $0 \leq s \leq r$ and D'_* the sequence D_1, \dots, D_s , we have the inclusion

$$\text{res}_{D_*/D'_*} : \mathcal{M}_*(X)_{D_*} \rightarrow \mathcal{M}_*(X)_{D'_*}$$

which defines the inclusion $\text{res}_{D_*/D'_*} : \mathcal{Z}_*(X)_{D_*} \rightarrow \mathcal{Z}_*(X)_{D'_*}$.

1.3. The dimension axiom. We define the group $\underline{\mathcal{Z}}_*(X)_{D_*}$.

Definition 1.6. Let X be in \mathbf{Sch}_k and let D_1, \dots, D_r be pseudo-divisors on X . Let $\langle \mathcal{R}_*^{Dim} \rangle(X)_{D_*}$ be the subgroup of $\mathcal{Z}_*(X)_{D_*}$ generated by cobordism cycles of the form

$$(f : Y \rightarrow X, \pi^*(L_1), \dots, \pi^*(L_m), M_1, \dots, M_s),$$

where $\pi : Y \rightarrow Z$ is a smooth quasi-projective morphism, Z is in \mathbf{Sm}_k , L_1, \dots, L_m are line bundles on Z and $m > \dim_k Z$. We set

$$\underline{\mathcal{Z}}_*(X)_{D_*} := \mathcal{Z}_*(X)_{D_*} / \langle \mathcal{R}_*^{Dim} \rangle(X)_{D_*}.$$

We have functoriality for smooth quasi-projective morphisms of relative dimension d , $f : X' \rightarrow X$:

$$\begin{aligned} f^* : \mathcal{M}(X)_{D_*} &\rightarrow \mathcal{M}(X')_{f^*(D_*)}; \quad f^*(g : Y \rightarrow X) := p_1 : X' \times_X Y \rightarrow X', \\ f^* : \mathcal{Z}_*(X)_{D_*} &\rightarrow \mathcal{Z}_{*+d}(X')_{f^*(D_*)}; \\ &\quad f^*(g, L_1, \dots, L_m) := (f^*g, p_2^*L_1, \dots, p_2^*L_m) \\ f^* : \underline{\mathcal{Z}}_*(X)_{D_*} &\rightarrow \underline{\mathcal{Z}}_{*+d}(X')_{f^*(D_*)}; \\ &\quad f^*(g, L_1, \dots, L_m) := (f^*g, p_2^*L_1, \dots, p_2^*L_m), \end{aligned}$$

and push-forward maps for projective morphisms $f : X' \rightarrow X$:

$$\begin{aligned} f_* : \mathcal{M}(X')_{f^*(D_*)} &\rightarrow \mathcal{M}(X)_{D_*}; & f_*(g : Y \rightarrow X') &:= f \circ g : Y \rightarrow X, \\ f_* : \mathcal{Z}_*(X')_{f^*(D_*)} &\rightarrow \mathcal{Z}_*(X)_{D_*}; & f_*(g, L_1, \dots, L_m) &:= (f_*(g), L_1, \dots, L_m), \\ f_* : \underline{\mathcal{Z}}_*(X')_{f^*(D_*)} &\rightarrow \underline{\mathcal{Z}}_*(X)_{D_*}; & f_*(g, L_1, \dots, L_m) &:= (f_*(g), L_1, \dots, L_m). \end{aligned}$$

Also, for $L \rightarrow X$ a line bundle on X , we have the Chern class endomorphism

$$\tilde{c}_1(L) : \mathcal{Z}_*(X)_{D_*} \rightarrow \mathcal{Z}_{*-1}(X)_{D_*},$$

defined as for $\mathcal{Z}_*(X)$:

$$\tilde{c}_1(L)((f : Y \rightarrow X, L_1, \dots, L_r)) := (f : Y \rightarrow X, L_1, \dots, L_r, f^*L).$$

This descends to the locally nilpotent endomorphism

$$\tilde{c}_1(L) : \underline{\mathcal{Z}}_*(X)_{D_*} \rightarrow \underline{\mathcal{Z}}_{*-1}(X)_{D_*}.$$

The operation of product over k defines external products

$$\times : \mathcal{Z}_*(X) \otimes \mathcal{Z}_*(X')_{D_*} \rightarrow \mathcal{Z}_*(X \times_k X')_{D_*},$$

which descend to $\underline{\mathcal{Z}}_*(-)$.

For $0 \leq s \leq r$ and for D'_* the subsequence D_1, \dots, D_s , the map $\text{res}_{D_*/D'_*} : \mathcal{Z}_*(X)_{D_*} \rightarrow \mathcal{Z}_*(X)_{D'_*}$ descends to a homomorphism

$$\text{res}_{D_*/D'_*} : \underline{\mathcal{Z}}_*(X)_{D_*} \rightarrow \underline{\mathcal{Z}}_*(X)_{D'_*}.$$

Remark 1.7. The homomorphism $\text{res}_{D_*/D'_*} : \underline{\mathcal{Z}}_*(X)_{D_*} \rightarrow \underline{\mathcal{Z}}_*(X)_{D'_*}$ is a split injection. Indeed, we have the identity

$$\langle \mathcal{R}_*^{Dim} \rangle(X)_{D_*} = \langle \mathcal{R}_*^{Dim} \rangle(X)_{D'_*} \cap \mathcal{Z}_*(X)_{D_*}$$

and both inclusions $\langle \mathcal{R}_*^{Dim} \rangle(X)_{D_*} \subset \mathcal{Z}_*(X)_{D_*}$, $\langle \mathcal{R}_*^{Dim} \rangle(X)_{D'_*} \subset \mathcal{Z}_*(X)_{D'_*}$ are inclusions of free subgroups on subsets of the generators of the free abelian groups $\mathcal{Z}_*(X)_{D_*}$, $\mathcal{Z}_*(X)_{D'_*}$. In particular, taking D'_* to be the empty set of pseudo-divisors, we may identify $\mathbb{L}_* \otimes \underline{\mathcal{Z}}_*(X)_{D_*}$ with this \mathbb{L}_* -submodule of $\mathbb{L}_* \otimes \underline{\mathcal{Z}}_*(X)$; this allows us to speak of the intersection $\mathbb{L}_* \otimes \underline{\mathcal{Z}}_*(X)_{D_*} \cap \mathbb{L}_* \otimes \underline{\mathcal{Z}}_*(X)_{D'_*}$ for two (or more) sequences of pseudo-divisors on X , the intersection taking place in $\mathbb{L}_* \otimes \underline{\mathcal{Z}}_*(X)$. We define intersections $\underline{\mathcal{Z}}_*(X)_{D_*} \cap \underline{\mathcal{Z}}_*(X)_{D'_*}$, $\mathcal{Z}_*(X)_{D_*} \cap \underline{\mathcal{Z}}_*(X)_{D'_*}$ or $\mathcal{M}_*(X)_{D_*} \cap \mathcal{M}_*(X)_{D'_*}$ similarly, the intersections taking place in $\underline{\mathcal{Z}}_*(X)$, $\mathcal{Z}_*(X)$ and $\mathcal{M}_*(X)$, respectively.

The maps res_{D_*/D'_*} are all natural with respect to the operations described above, in particular, taking D'_* to be the empty set of pseudo-divisors, the operations on $\mathcal{M}_*(-)_{D_*}$, $\mathcal{Z}_*(-)_{D_*}$ and $\underline{\mathcal{Z}}_*(-)_{D_*}$ are compatible with the corresponding ones on \mathcal{M}_* , \mathcal{Z}_* and $\underline{\mathcal{Z}}_*$ via $\text{res}_{D_*/\emptyset}$. Consequently, all relations and compatibilities among these operations that hold in $\mathcal{M}(-)$, $\mathcal{Z}_*(-)$ or $\underline{\mathcal{Z}}_*(-)$ hold for $\mathcal{M}(-)_{D_*}$, $\mathcal{Z}_*(-)_{D_*}$ or $\underline{\mathcal{Z}}_*(-)_{D_*}$ as well.

1.4. Good position. We define a notion of “good position” of a divisor E with respect to a sequence of pseudo-divisors D_1, \dots, D_r .

Definition 1.8. Let Y be in \mathbf{Sm}_k , irreducible, and let D_1, \dots, D_r be pseudo-divisors on Y such that Id_Y is in $\mathcal{M}(Y)_{D_*}$. Let E be an effective simple normal crossing divisor on Y . We say that E is *in good position with respect to D_** if either

- i) Id_Y has no leading pseudo-divisor,
- or
- ii) Id_Y has leading pseudo-divisor D_{i_0} and $E + \text{div} D_{i_0}$ is a simple normal crossing divisor on Y .

We extend this notion to Cartier divisors E on a not necessarily irreducible $Y \in \mathbf{Sm}_k$ by requiring that $E \cap Y_i$ is in good position with respect to D_* for each irreducible component Y_i of Y .

Remarks 1.9. (1) For $f : Y \rightarrow X$ in $\mathcal{M}(X)_{D_*}$ with E a simple normal crossing divisor on Y , if E is in good position with respect to D_* , then E is admissible with respect to D_* ; this is lemma 1.3.

(2) If E is in good position with respect to D_* on some $Y \in \mathbf{Sm}_k$ and C is an effective Cartier divisor with $|C| \subset |E|$, then C is also in good position with respect to D_* .

Lemma 1.10. Let D, D_1, \dots, D_r be pseudo-divisors on X , let $f : Y \rightarrow X$ be morphism with $Y \in \mathbf{Sm}_k$, Y irreducible, and with f in $\mathcal{M}(X)_{D, D_*}$. Let E be a simple normal crossing divisor on Y , in good position with respect to D, D_* . Suppose that $f(Y) \not\subset |D|$. Then

(1) $\text{div} f^* D$ is a simple normal normal crossing divisor on Y , admissible for D_* and in good position with respect to D, D_* .

(2) For each irreducible component F of a face of $\text{div} f^* D$, if $F \not\subset |E|$, then $F \cap E$ is a simple normal crossing divisor on F , in good position with respect to D_* .

(3) Let F be an irreducible component of a face of $E + \text{div} f^* D$ with $F \subset |f^* D|$. Then the inclusion $F \rightarrow Y$ is in $\mathcal{M}(Y)_{D_*}$.

(4) Suppose that E is smooth over k , and let $F_1 \subset F_2$ be irreducible components of faces of $\text{div} f^* D$, with $F_1 \not\subset |E|$ and F_1 a codimension one closed subscheme of F_2 . Then for $i = 1, 2$, $F_i \cap E$ is smooth, the inclusion $F_i \cap E \rightarrow Y$ is in $\mathcal{M}(Y)_{D_*}$ and the divisor $F_1 \cap E$ on $F_2 \cap E$ is in good position with respect to D_* .

(5) Let $F_1 \subset F_2$ be the inclusion of irreducible components of faces of E , with F_1 a codimension one closed subscheme of F_2 . Then the divisor F_1 on F_2 is in good position with respect to D, D_* .

(6) Let C and S be effective Cartier divisors on Y with support contained in E . Suppose that S is smooth and C and S have no common components.

Then $S \rightarrow Y \rightarrow X$ is in $\mathcal{M}(X)_{D,D_*}$ and the simple normal crossing divisor $C \cap S$ on S is in good position with respect to D, D_* .

Proof. We write D_0 for D when convenient. Under our assumptions, D is the leading pseudo-divisor for f with respect to D, D_1, \dots, D_r . By lemma 1.4, $\text{div} f^*D$ is a simple normal crossing divisor on Y , admissible with respect to D_* . As $\text{div} f^*D + \text{div} f^*D$ is a simple normal crossing divisor on Y , $\text{div} f^*D$ is in good position with respect to D, D_* , proving (1).

For (2), take F to be an irreducible component of a face of $\text{div} f^*D$. Since $E + \text{div} f^*D$ is a simple normal crossing divisor on Y , it follows that $F \cap E$ is a simple normal crossing divisor on F as long as $F \not\subset |E|$. Let $g : F \rightarrow X$ be the composition $f \circ i_F$, where $i_F : F \rightarrow Y$ is the inclusion.

As $\text{div} f^*D$ is admissible for D_* , g is in $\mathcal{M}(X)_{D_*}$. Supposing that g has no leading pseudo-divisor with respect to D_* , it follows that i_F^*E is in good position with respect to D_* . Suppose then that D_{i_0} is the leading pseudo-divisor for g with respect to D_* . Then $F \subset \text{div} f^*D_i$ for all i , $0 \leq i < i_0$ and thus $\text{div} g^*D_{i_0} = i_F^*(\cap_{i=0}^{i_0} \text{div} f^*D_i)$. But since Id_Y is in $\mathcal{M}(Y)_{D,D_*}$, $\cap_{i=0}^{i_0} \text{div} f^*D_i$ is a Cartier divisor on Y and is a subdivisor of $\text{div} f^*D = \text{div} f^*D_0$. Thus $\cap_{i=0}^{i_0} \text{div} f^*D_i + E$ is a simple normal crossing divisor on Y and is a subdivisor of $\text{div} f^*D + E$. As F is a component of a face of $\text{div} f^*D + E$, it follows that $i_F^*(E + \text{div} f^*D_{i_0})$ is a simple normal crossing divisor on F , and thus i_F^*E is in good position with respect to D_* .

The assertion (3) follows from remark 1.9(1): $E + \text{div} f^*D$ is in good position with respect to D, D_* , hence each irreducible component F of a face of $E + \text{div} f^*D$ is admissible with respect to D, D_* . If F is contained in $|f^*D|$, then D cannot be the leading pseudo-divisor for $F \rightarrow Y$, hence $F \rightarrow Y$ is in $\mathcal{M}(Y)_{D_*}$.

For (4), the fact that $F_i \cap E$ is smooth and $F_1 \cap E$ has codimension one on $F_2 \cap E$ follows from the fact that $E + \text{div} f^*D$ is a simple normal crossing divisor. The inclusions $F_i \cap E \rightarrow Y$ are in $\mathcal{M}(Y)_{D_*}$ by (3). If D_{i_0} is the leading pseudo-divisor for $F_2 \cap E \rightarrow Y$, then $E + \cap_{i=0}^{i_0} \text{div} f^*D_i$ is a simple normal crossing divisor on Y and a sub-divisor of $E + \text{div} f^*D$. It follows that the divisor

$$F_1 \cap E + \text{div} f^*D_{i_0} \cap F_2 \cap E = F_1 \cap E + \cap_{i=0}^{i_0} \text{div} f^*D_i \cap F_2 \cap E$$

on $F_2 \cap E$ is a simple normal crossing divisor.

For (5), Id_{F_2} is in $\mathcal{M}(F_2)_{D,D_*}$, since E is admissible for D, D_* by remark 1.9(1). Suppose that D_{i_0} is the leading pseudo-divisor for F_2 with respect to D, D_* . As above, $\text{div} f^*D_{i_0} \cap F_2$ is equal to $C \cap F_2$ for C the subdivisor $\cap_{i=0}^{i_0} \text{div} f^*D_i$ of $\text{div} D$. $F_1 \subset F_2$ is an irreducible component of $F_2 \cap E_i$ for E_i some irreducible component of E . As $E_i + C$ is a subdivisor of the simple normal crossing divisor $E + \text{div} D$, the intersection $F_2 \cap (E_i + C)$ is a simple normal crossing divisor on F_2 . As this intersection contains $F_1 + F_2 \cap \text{div} D_{i_0}$ as a subdivisor, it follows that $F_1 + F_2 \cap \text{div} D_{i_0}$ is also a simple normal crossing divisor on F_2 , and thus F_1 is in good position on F_2 .

with respect to D, D_* . In case there is no leading pseudo-divisor for Id_{F_2} , then F_1 is in good position on F_2 with respect to D, D_* , as F_1 is a smooth divisor on F_2 .

Finally, for (6), $C + S$ is a simple normal crossing divisor and C and S have no common components, hence $C \cap S$ is a simple normal crossing divisor on S . Since S is a disjoint union of faces of E , $S \rightarrow X$ is in $\mathcal{M}(X)_{D, D_*}$ by remark 1.9(1). Let S' be an irreducible component of S ; we need to check that $S' \cap C$ is in good position with respect to D, D_* . Suppose D is the leading pseudo-divisor for $S' \rightarrow X$. Then $C + \text{div} f^* D + S'$ is a simple normal crossing divisor on Y and S' has no components in common with $C + \text{div} f^* D$, so $S' \cap (C + \text{div} f^* D)$ is a simple normal crossing divisor on S' and thus $C \cap S'$ is in good position on S' with respect to D, D_* . If D_{i_0} is the leading pseudo-divisor for $S' \rightarrow X$ for some $i_0 \geq 1$, then $S' \subset |f^* D_{i_0}|$ for $0 \leq i < i_0$ and $S' \cap \text{div} f^* D_{i_0} = S' \cap \bigcap_{i=0}^{i_0} \text{div} f^* D_i$. Since $Y \rightarrow X$ is in $\mathcal{M}(X)_{D, D_*}$, $B := \bigcap_{i=0}^{i_0} \text{div} f^* D_i$ is a subdivisor of $\text{div} f^* D$ and thus as above, $S' \cap (C + B)$ is a simple normal crossing divisor on S' . Thus, $C \cap S'$ is in good position on S' with respect to D, D_* in this case as well. In case $S' \rightarrow X$ has no leading pseudo-divisor, then $C \cap S'$ is trivially in good position with respect to D, D_* . \square

1.5. Refined algebraic cobordism. We complete the definition of $\Omega_*(-)_{D_*}$.

Definition 1.11. Let X be in \mathbf{Sch}_k and let D_1, \dots, D_r be pseudo-divisors on X . Let $\langle \mathcal{R}_*^{\text{Sect}} \rangle(X)_{D_*}$ be the subgroup of $\underline{\mathcal{Z}}_*(X)_{D_*}$ generated by elements of the form

$$[f : Y \rightarrow X, L_1, \dots, L_m] - [f \circ i : Z \rightarrow X, i^*(L_1), \dots, i^*(L_{m-1})],$$

with $m > 0$, $[f : Y \rightarrow X, L_1, \dots, L_m]$ a cobordism cycle in $\mathcal{Z}_*(X)_{D_*}$ and $i : Z \rightarrow Y$ the closed immersion of the smooth subscheme defined by the vanishing of a section $s : Y \rightarrow L_m$ transverse to the zero-section, such that Z is in good position with respect to D_* .

We define $\underline{\Omega}_*(X)_{D_*}$ by

$$\underline{\Omega}_*(X)_{D_*} := \underline{\mathcal{Z}}_*(X)_{D_*} / \langle \mathcal{R}_*^{\text{Sect}} \rangle(X)_{D_*}.$$

Let $0 \leq s \leq r$ and let D'_* be the sequence D_1, \dots, D_s . The map $\text{res}_{D_*/D'_*} : \underline{\mathcal{Z}}_*(X)_{D_*} \rightarrow \underline{\mathcal{Z}}_*(X)_{D'_*}$ descends to $\text{res}_{D_*/D'_*} : \underline{\Omega}_*(X)_{D_*} \rightarrow \underline{\Omega}_*(X)_{D'_*}$. The operations f^*, f_* and $\tilde{c}_1(L)$, as well as the external products also descend to the quotient $\underline{\Omega}_*(-)_{D_*}$ of $\underline{\mathcal{Z}}_*(-)_{D_*}$ and the maps res_{D_*/D'_*} are natural with respect to the operations and products.

We have the universal formal group law $(F_{\mathbb{L}}, \mathbb{L}_*)$ with coefficient ring \mathbb{L}_* the Lazard ring. If $T_1, T_2 : B \rightarrow B$ are commuting locally nilpotent operators on an abelian group B , and $F(u, v) = \sum_{i,j} a_{ij} u^i v^j$ is a power series with \mathbb{L}_* -coefficients, we have the well-defined \mathbb{L}_* -linear operator $F(T_1, T_2) : \mathbb{L}_* \otimes B \rightarrow \mathbb{L}_* \otimes B$ defined by

$$F(T_1, T_2)(a \otimes b) := \sum_{i,j} a a_{ij} \otimes (T_1^i \circ T_2^j)(b).$$

Definition 1.12. For X in \mathbf{Sch}_k , let $\langle \mathcal{R}_*^{FGL} \rangle(X)_{D_*}$ be the \mathbb{L}_* -submodule of $\mathbb{L}_* \otimes \underline{\Omega}_*(X)_{D_*}$ generated by elements of the form

$$(\mathrm{Id} \otimes f_*)(F_{\mathbb{L}}(\tilde{c}_1(L), \tilde{c}_1(M))(\eta) - \tilde{c}_1(L \otimes M)(\eta)),$$

where $f : Y \rightarrow X$ is in $\mathcal{M}(X)_{D_*}$, L and M are line bundles on Y , and η is in $\underline{\Omega}_*(Y)_{f^*(D_*)}$. We set

$$\Omega_*(X)_{D_*} := \mathbb{L}_* \otimes \underline{\Omega}_*(X)_{D_*} / \langle \mathcal{R}_*^{FGL} \rangle(X)_{D_*}.$$

For $0 \leq s \leq r$ and D'_* the sequence D_1, \dots, D_s , the natural transformation $\mathrm{res}_{D_*/D'_*} : \underline{\Omega}_*(X)_{D_*} \rightarrow \underline{\Omega}_*(X)_{D'_*}$ descends to an \mathbb{L}_* -linear transformation $\mathrm{res}_{D_*/D'_*} : \Omega_*(X)_{D_*} \rightarrow \Omega_*(X)_{D'_*}$. The operations we have defined for $\underline{\Omega}_*(-)_-$: f^* , f_* , $\tilde{c}_1(L)$ and external products, all descend to \mathbb{L}_* -linear or bi-linear operations on $\Omega_*(-)_-$, and the maps res_{D_*/D'_*} are natural with respect to these operations.

If $f : Y \rightarrow X$ is an X -scheme, and D_1, \dots, D_r are pseudo-divisors on X , we will often write $\Omega_*(Y)_{D_*}$ for $\Omega_*(Y)_{f^*(D_*)}$, and similarly for $\underline{\Omega}_*(Y)_{D_*}$.

1.6. Refined divisor classes. The operators

$$\tilde{c}_1(L) : \Omega_*(X)_{D_*} \rightarrow \Omega_{*-1}(X)_{D_*}$$

are locally nilpotent and commute with one another, thus, if we have line bundles L_1, \dots, L_m on X , and a power series $F(u_1, \dots, u_m)$ with \mathbb{L}_* -coefficients (of total degree d), we have the \mathbb{L}_* -linear endomorphism

$$F(\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_m)) : \Omega_*(X)_{D_*} \rightarrow \Omega_{*+d}(X)_{D_*}.$$

This lifts to the level of $\mathbb{L}_* \otimes \underline{\Omega}_*(X)_{D_*}$, giving us the \mathbb{L}_* -linear endomorphism

$$\underline{F}(\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_m)) : \mathbb{L}_* \otimes \underline{\Omega}_*(X)_{D_*} \rightarrow \mathbb{L}_* \otimes \underline{\Omega}_{*+d}(X)_{D_*}.$$

If we have a morphism $f : X' \rightarrow X$ and an element $\eta \in \Omega_*(X')_{D_*}$, we often write $F(\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_m))(\eta)$ for $F(\tilde{c}_1(f^*L_1), \dots, \tilde{c}_1(f^*L_m))(\eta)$, when the context makes the meaning clear; we similarly write $\underline{F}(\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_m))(\underline{\eta})$ for $\underline{F}(\tilde{c}_1(f^*L_1), \dots, \tilde{c}_1(f^*L_m))(\underline{\eta})$ for an $\underline{\eta} \in \mathbb{L}_* \otimes \underline{\Omega}_*(X')_{D_*}$.

If $f : Y \rightarrow X$ is in $\mathcal{M}(X)_{D_*}$, we have the element $1_Y^{D_*} = [\mathrm{Id} : Y \rightarrow Y] \in \mathcal{M}_*(Y)_{D_*}$. Given line bundles L_1, \dots, L_m on Y we define

$$[Y; F(L_1, \dots, L_m)]_{D_*} := \underline{F}(\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_m))(1_Y^{D_*}) \in \mathbb{L}_* \otimes \underline{\Omega}_*(Y)_{D_*}$$

and

$$[Y; F(L_1, \dots, L_r)]_{D_*} := F(\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_r))(1_Y^{D_*}) \in \Omega_*(Y)_{D_*}.$$

As above, if $f : Y' \rightarrow Y$ is a morphism, we often write $[Y'; F(L_1, \dots, L_m)]_{D_*}$ for $[Y'; F(f^*L_1, \dots, f^*L_m)]_{D_*}$ and similarly write $[Y'; F(L_1, \dots, L_r)]_{D_*}$ for $[Y'; F(f^*L_1, \dots, f^*L_r)]_{D_*}$.

We recall some notation from [6, §3.1.1]. Let n_1, \dots, n_m be non-negative integers. We have the power series with \mathbb{L}_* -coefficients F^{n_1, \dots, n_m} giving the sum in the universal group law $(F_{\mathbb{L}}, \mathbb{L}_*)$:

$$F^{n_1, \dots, n_m}(u_1, \dots, u_m) = n_1 \cdot_F u_1 +_F \dots +_F n_m \cdot_F u_m.$$

We have the canonical decomposition

$$F^{n_1, \dots, n_m}(u_1, \dots, u_m) = \sum_J u^J F_J^{n_1, \dots, n_m}(u_1, \dots, u_m).$$

Here $J = (j_1, \dots, j_m) \in \{0, 1\}^m$, $u^J := u_1^{j_1} \dots u_m^{j_m}$ and $F_J^{n_1, \dots, n_m}$ is a power series with \mathbb{L} -coefficients that only involves the u_i with $j_i = 1$; this property and the above identity uniquely characterize the $F_J^{n_1, \dots, n_m}$.

If $E = \sum_{i=1}^m n_i E_i$ is a simple normal crossing divisor on a scheme $Y \in \mathbf{Sm}_k$, with support $|E|$ and irreducible components E_1, \dots, E_m , we have for $J \in \{0, 1\}^m$ the face $E^J := \cap_{j_i=1} E_i$, inclusions $\iota^J : E^J \rightarrow |E|$, $i_J : E^J \rightarrow Y$, and line bundles $O_Y(E_i)^J := i_J^*(O_Y(E_i))$ on E^J . We have defined the divisor class $[E \rightarrow |E|]$ of $\Omega_*(|E|)$ by the formula (see [6, definition 3.1.5])

$$[E \rightarrow |E|] := \sum_J \iota_*^J([E^J; F_J^{n_1, \dots, n_m}(O_Y(E_1)^J, \dots, O_Y(E_m)^J)]).$$

Suppose now that we have $f : Y \rightarrow X$ in $\mathcal{M}(X)_{D_*}$, and a simple normal crossing divisor E on Y , such that E is admissible with respect to D_* . Write $E = \sum_{i=1}^m n_i E_i$, with the E_i irreducible, as above.

Since E is admissible with respect to D_* , Id_{E^J} is in $\mathcal{M}(E^J)_{D_*}$ for each index J , so we have the class

$$[E^J; F_J^{n_1, \dots, n_m}(O_Y(E_1)^J, \dots, O_Y(E_m)^J)]_{D_*} \in \Omega_*(E^J)_{D_*},$$

giving the *refined divisor class*

$$[E \rightarrow |E|]_{D_*} := \sum_J \iota_*^J([E^J; F_J^{n_1, \dots, n_m}(O_Y(E_1)^J, \dots, O_Y(E_m)^J)]_{D_*})$$

in $\Omega_*(|E|)_{D_*}$. In fact, the same formula gives us a well-defined class

$$\underline{[E \rightarrow |E|]}_{D_*} := \sum_J \iota_*^J \underline{[E^J; F_J^{n_1, \dots, n_m}(O_Y(E_1)^J, \dots, O_Y(E_m)^J)]}_{D_*}$$

in $\mathbb{L}_* \otimes \underline{\mathcal{Z}}_*(|E|)_{D_*}$ lifting $[E \rightarrow |E|]_{D_*}$.

If we have a pseudo-divisor \tilde{E} on Y such that $E := \text{div} \tilde{E}$ is a Cartier divisor on Y , admissible with respect to D_* , we often write $[\tilde{E} \rightarrow |\tilde{E}|]_{D_*}$ for $[E \rightarrow |E|]_{D_*}$; if $f : |E| \rightarrow X'$ is a projective morphism of finite type X -schemes, we write $[E \rightarrow X']_{D_*}$ for $f_*([E \rightarrow |E|]_{D_*})$ and $\underline{[E \rightarrow X']}_{D_*}$ for $f_* \underline{[E \rightarrow |E|]}_{D_*}$. These divisor classes and those for a truncated sequence $D'_* := D_1, \dots, D_s$ are compatible via the forget maps res_{D_*/D'_*} .

Lemma 1.13. *Let D_1, \dots, D_r be pseudo-divisors on some $Y \in \mathbf{Sm}_k$ and let E be a simple normal crossing divisor on Y , admissible with respect to D_* . Write $E = \sum_{i=1}^m n_i E_i$ with each E_i smooth, but not necessarily irreducible. We assume that E_i and E_j have no common components if $i \neq j$. For each index $J = (j_1, \dots, j_m) \in \{0, 1\}^m$ let $E^J = \cap_{j_i=1} E_i$, with inclusions $\iota^J : E^J \rightarrow |E|$, $i_J : E^J \rightarrow Y$. Let $L_i = O_Y(E_i)$, $L_i^J = i_J^* L_i$. Then*

$$[E \rightarrow |E|]_{D_*} = \sum_J \iota_*^J([E^J; F_J^{n_1, \dots, n_m}(L_1^J, \dots, L_m^J)]_{D_*}).$$

in $\Omega_*(|E|)_{D_*}$.

Proof. For each i , write $E_i = \coprod_{j=1}^{p_i} E_{ij}$, with E_{ij} irreducible (and smooth). Then $E = \sum_{i=1}^m \sum_{j=1}^{p_i} n_i E_{ij}$. Let $n_i^{(p_i)}$ be the sequence n_i, \dots, n_i , with p_i terms. For $J_* = (J_1, \dots, J_m) \in \{0, 1\}^{p_1} \times \dots \times \{0, 1\}^{p_m}$, $J_i = (j_{i1}, \dots, j_{ip_i})$, we have the corresponding face E^{J_*} of E ,

$$E^{J_*} = \cap_{i=1}^m \cap_{j_{ik}=1} E_{ik}.$$

We let $u_{i*} = u_{i1}, \dots, u_{ip_i}$. The formal group law sum

$$\begin{aligned} & F^{n_1^{(p_1)}, \dots, n_m^{(p_m)}}(u_{1*}, \dots, u_{m*}) \\ &:= n_1 \cdot_F (u_{11} +_F \dots +_F u_{1p_1}) +_F \dots +_F n_m \cdot_F (u_{m1} +_F \dots +_F u_{mp_m}) \end{aligned}$$

decomposes following our usual conventions as

$$F^{n_1^{(p_1)}, \dots, n_m^{(p_m)}}(u_{1*}, \dots, u_{m*}) = \sum_{J_*} u^{J_*} F_{J_*}^{n_1^{(p_1)}, \dots, n_m^{(p_m)}}(u_{1*}, \dots, u_{m*}),$$

where $u^{J_*} = \prod_{ij} u_{ik}^{j_{ik}}$. We let $L_{ij} = O_Y(E_{ij})$, $L_{ij}^{J_*}$ the restriction of L_{ij} to E^{J_*} , and write $L_{i*}^{J_*}$ for the sequence $L_{i1}^{J_*}, \dots, L_{ip_i}^{J_*}$ and let $\iota^{J_*} : E^{J_*} \rightarrow |E|$ be the inclusion. With these notations, the divisor class $[E \rightarrow |E|]_{D_*}$ is

$$[E \rightarrow |E|]_{D_*} = \sum_{J_*} \iota_*^{J_*} ([E^{J_*}; F_{J_*}^{n_1^{(p_1)}, \dots, n_m^{(p_m)}}(L_{1*}^{J_*}, \dots, L_{m*}^{J_*})]_{D_*}).$$

Fix an index $J = (j_1, \dots, j_m) \in \{0, 1\}^m$. The divisor E_i thus contains E^J if and only if $j_i = 1$.

To each $J = (j_1, \dots, j_m)$ as above, the closed subscheme E^J breaks up as a disjoint union of certain faces E^{J_*} , namely, exactly for those $J_* = (J_1, \dots, J_m)$ such that, if $j_i = 0$, then $J_i = (0, \dots, 0) = 0^{p_i}$, and if $j_i = 1$, then the index $J_i \in \{0, 1\}^{p_i}$ contains exactly one 1; if this 1 appears in the q th spot, we write this as $J_i = e_q^{(p_i)}$. Since $E_{ij} \cap E_{ij'} = \emptyset$ for $j \neq j'$, the face E^{J_*} is empty if one J_i contains more than one 1, and thus E^J is the disjoint union of the faces E^{J_*} with $J_i = 0^{p_i}$ if $j_i = 0$ and $J_i = e_{q_i}^{(p_i)}$ for some q_i with $1 \leq q_i \leq p_i$ if $j_i = 1$. For such a J_* , we set $q_i = 0$ if $j_i = 0$, set $e_0^{(p_i)} = 0^{p_i}$, set $q_* = (q_1, \dots, q_m)$, and write the index $J_* = (e_{q_1}^{(p_1)}, \dots, e_{q_m}^{(p_m)})$ as $J_* = J(q_*)$. Let $S(J) \subset \prod_{i=1}^m \{0, \dots, p_i\}$ be the subset consisting of those q_* such that $q_i = 0$ if and only if $j_i = 0$. In this notation, we have

$$E^J = \coprod_{q_* \in S(J)} E^{J(q_*)}.$$

Take $q_* \in S(J)$. We claim that for each i such that $j_i = 1$,

$$\tilde{c}_1(L_i)(1_{E^{J(q_*)}}^{D_*}) = \tilde{c}_1(L_{iq_i})(1_{E^{J(q_*)}}^{D_*})$$

in $\Omega_*(E^{J(q_*)})_{D_*}$. Indeed, $E_{ij} \cap E^{J(q_*)} = \emptyset$ for all $j \neq q_i$. Applying the relations $\langle \mathcal{R}_*^{Sect} \rangle(E^{J(q_*)})_{D_*}$ for the empty divisor, we have $\tilde{c}_1(L_{ij})(1_{E^{J(q_*)}}^{D_*}) =$

0 for all $j \neq q_i$. Noting that $L_i = \otimes_{j=1}^{p_i} L_{ij}$ and applying the formal group relations $\langle \mathcal{R}_*^{FGL} \rangle (E^{J(q_*)})_{D_*}$ we have

$$\tilde{c}_1(L_i)(1_{E^{J(q_*)}}^{D_*}) = F^{1, \dots, 1}(\tilde{c}_1(L_{i1}), \dots, \tilde{c}_1(L_{ip_i}))(1_{E^{J(q_*)}}^{D_*}) = \tilde{c}_1(L_{iq_i})(1_{E^{J(q_*)}}^{D_*}).$$

Since $F_J^{n_1, \dots, n_m}(u_1, \dots, u_m)$ does not involve u_i if $j_i = 0$, this gives the relation

$$\begin{aligned} F_J^{n_1, \dots, n_m}(\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_m))(1_{E^{J(q_*)}}^{D_*}) \\ = F_{J(q_*)}^{n_{1*}, \dots, n_{m*}}(\tilde{c}_1(L_{1*}), \dots, \tilde{c}_1(L_{m*}))(1_{E^{J(q_*)}}^{D_*}) \end{aligned}$$

in $\Omega_*(E^{J(q_*)})_{D_*}$. Therefore

$$\begin{aligned} \sum_J \iota_*^J([E^J; F_J^{n_1, \dots, n_m}(L_1^J, \dots, L_m^J)]_{D_*}) \\ = \sum_J \iota_*^J(F_J^{n_1, \dots, n_m}(\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_m))(1_{E^J}^{D_*})) \\ = \sum_J \sum_{q \in S(J)} \iota_*^{J(q_*)}(F_J^{n_1, \dots, n_m}(\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_m))(1_{E^{J(q_*)}}^{D_*})) \\ = \sum_J \sum_{q \in S(J)} \iota_*^{J(q_*)}(F_{J(q_*)}^{n_{1*}, \dots, n_{m*}}(\tilde{c}_1(L_{1*}), \dots, \tilde{c}_1(L_{m*}))(1_{E^{J(q_*)}}^{D_*})) \\ = \sum_{J_*} \iota_*^{J_*}(F_{J_*}^{n_{1*}, \dots, n_{m*}}(\tilde{c}_1(L_{1*}), \dots, \tilde{c}_1(L_{m*}))(1_{E^{J_*}}^{D_*})) \\ = [E \rightarrow |E|]_{D_*}. \end{aligned}$$

□

Lemma 1.14. *Let E be a simple normal crossing divisor on some $Y \in \mathbf{Sm}_k$. Let D_1, \dots, D_r be pseudo-divisors on Y . Suppose that Id_Y is in $\mathcal{M}(Y)_{D_*}$ and that E is in good position with respect to D_* . Then*

$$[E \rightarrow Y]_{D_*} = [Y; O_Y(E)]_{D_*}$$

in $\Omega_*(Y)_{D_*}$.

Proof. Write $E = \sum_{i=1}^m n_i E_i$ with the E_i irreducible and let $i : |E| \rightarrow Y$ be the inclusion. For $J = (j_1, \dots, j_m) \in \{0, 1\}^m$, let E^J be the corresponding face of E with inclusions $\iota^J : E^J \rightarrow |E|$, $i^J : E^J \rightarrow Y$. If $j_i = 1$ for some i , let $J' = (j_1, \dots, j_i - 1, \dots, j_m)$. Then by lemma 1.10(5), the subscheme E^J of $E^{J'}$ is a divisor in good position with respect to D_* . Repeated applications of the relations $\langle \mathcal{R}_*^{Sect} \rangle (-)_{D_*}$ thus give us the relation

$$i_*^J \left(F(\tilde{c}_1(L_1^J), \dots, \tilde{c}_1(L_m^J))(1_{E^J}^{D_*}) \right) = (\tilde{c}_1(L_*)^J F(\tilde{c}_1(L_1^J), \dots, \tilde{c}_1(L_m^J)))(1_Y^{D_*})$$

for arbitrary power series $F(u_1, \dots, u_m) \in \mathbb{L}_*[[u_1, \dots, u_m]]$ and line bundles L_1, \dots, L_m on Y , where $L_i^J := i_*^J L_i$ and $\tilde{c}_1(L_*)^J := \tilde{c}_1(L_1)^{j_1} \circ \dots \circ \tilde{c}_1(L_m)^{j_m}$.

Applying this to the definition of $[E \rightarrow |E|]_{D_*}$ gives us

$$\begin{aligned}
[E \rightarrow Y]_{D_*} &= i_*([E \rightarrow |E|]_{D_*}) \\
&= i_*\left(\sum_J \iota_*^J(F_J^{n_1, \dots, n_m}(\tilde{c}_1(O_Y(E_1))^J, \dots, \tilde{c}_1(O_Y(E_m))^J))(1_{E^J}^{D_*})\right) \\
&= \sum_J (\tilde{c}_1(O_Y(E_*))^J \circ F_J^{n_1, \dots, n_m}(\tilde{c}_1(O_Y(E_1)), \dots, \tilde{c}_1(O_Y(E_m))))(1_Y^{D_*}) \\
&= F^{n_1, \dots, n_m}(\tilde{c}_1(O_Y(E_1)), \dots, \tilde{c}_1(O_Y(E_m)))(1_Y^{D_*}) \\
&= \tilde{c}_1(O_Y(E))(1_Y^{D_*}) = [Y; O_Y(E)]_{D_*}.
\end{aligned}$$

□

Lemma 1.15. *Take Y in \mathbf{Sm}_k with pseudo-divisors D_1, \dots, D_r on Y such that Id_Y is in $\mathcal{M}(Y)_{D_*}$. Let Z_1, Z_2 be smooth disjoint divisors on Y .*

(1) *Assume that at least one of Z_1, Z_2 is in good position with respect to D_* . Then*

$$\tilde{c}_1(O_Y(Z_1)) \circ \tilde{c}_1(O_Y(Z_2))(1_Y^{D_*}) = 0$$

in $\Omega_*(Y)_{D_*}$.

(2) *Let D be an effective Cartier divisor on Y , let C be an effective Cartier divisor on Y with support contained in $|D|$ and let $i : |C| \rightarrow Y$ be the inclusion. Suppose that Id_Y is in $\mathcal{M}(Y)_{D, D_*}$ and that $Z_1 + Z_2$ is in good position with respect to D, D_* . Then $\tilde{c}_1(O_Y(Z_1)) \circ \tilde{c}_1(O_Y(Z_2))(1_C^{D_*}) = 0$ in $\Omega_*(C^J)_{D_*}$ for each face C^J of C , and*

$$\tilde{c}_1(i^*O_Y(Z_1)) \circ \tilde{c}_1(i^*O_Y(Z_2))([C \rightarrow |C|]_{D_*}) = 0$$

in $\Omega_*(|C|)_{D_*}$.

Proof. We first prove (2), assuming (1). From the hypotheses in (2) it follows that D is the leading pseudo-divisor for Id_Y with respect to D, D_* . By lemma 1.4, C is a simple normal crossing divisor on Y , admissible with respect to D_* ; in particular, the divisor class $[C \rightarrow |C|]_{D_*}$ is defined.

Write $C = \sum_{j=1}^m n_j C_j$ with each C_i irreducible. For each $J \in \{0, 1\}^m$, write C^J as a disjoint union of irreducible components, $C^J = \coprod_j C_j^J$, and let $\iota_j^J : C_j^J \rightarrow |C|$, $i_j^J : C_j^J \rightarrow Y$ be the inclusions. Then each inclusion $C_j^J \rightarrow Y$ is in $\mathcal{M}(Y)_{D_*}$, so the unit elements $1_{C_j^J}^{D_*} \in \mathcal{M}(C_j^J)_{D_*}$ are all defined.

Let $\eta_{J,j} = \tilde{c}_1(i_j^{J*}O_Y(Z_1)) \circ \tilde{c}_1(i_j^{J*}O_Y(Z_2))(1_{C_j^J}^{D_*})$. In $\Omega_*(|C|)_{D_*}$ we have the identity

$$\begin{aligned}
&\tilde{c}_1(i^*O_Y(Z_1)) \circ \tilde{c}_1(i^*O_Y(Z_2))([C \rightarrow |C|]_{D_*}) \\
&= \sum_{J,j} F_J^{n_1, \dots, n_m}(\tilde{c}_1(i^*O_X(C_1)), \dots, \tilde{c}_1(i^*O_Y(C_m)))(\iota_{j*}^J(\eta_{J,j})).
\end{aligned}$$

If C_j^J is not contained in $Z_1 \cup Z_2$, then $Z_1 \cap C_j^J$ and $Z_2 \cap C_j^J$ are smooth disjoint divisors on C_j^J ; by lemma 1.10(2), these are both in good position with respect to D_* . Thus (1) (for $Y = C_j^J$) implies that $\eta_{J,j} = 0$.

If C_j^J is contained in say Z_1 , then $C_j^J \cap Z_2 = \emptyset$ and thus $i_j^* O_Y(Z_2) \cong O_{C_j^J}$. Using the relations $\langle \mathcal{R}_*^{Sect} \rangle (C_j^J)_{D_*}$ in the case of an empty divisor, we see that $\tilde{c}_1(i_j^* O_Y(Z_2))(1_{C_j^J}^{D_*}) = 0$, so $\eta_{J,j} = 0$ in this case as well, and (2) follows.

For (1), let $i_j : Z_j \rightarrow Y$ be the inclusion. Since $\tilde{c}_1(O_Y(Z_1))$ and $\tilde{c}_1(O_Y(Z_2))$ commute, we may assume that Z_2 is in good position with respect to D_* . Using the relations $\langle \mathcal{R}_*^{Sect} \rangle (Y)_{D_*}$ gives

$$\tilde{c}_1(O_Y(Z_1)) \circ \tilde{c}_1(O_Y(Z_2))(1_Y^{D_*}) = i_{Z_2*} \left(\tilde{c}_1(i_2^* O_X(Z_1))(1_{Z_2}^{D_*}) \right)$$

in $\Omega_*(Y)_{D_*}$. Using $\langle \mathcal{R}_*^{Sect} \rangle (Z_2)_{D_*}$, again in the case of an empty divisor, gives $\tilde{c}_1(i_2^* O_X(Z_1))(1_{Z_2}^{D_*}) = 0$ in $\Omega_*(Z_2)_{D_*}$. \square

Lemma 1.16. *Let $f : X \rightarrow Z$ be a morphism in \mathbf{Sch}_k with Z in \mathbf{Sm}_k , and let L_1, \dots, L_m be line bundles on Z with $m > \dim_k Z$. Let D_1, \dots, D_r be pseudo-divisors on X . Then the operator $\tilde{c}_1(f^* L_1) \circ \dots \circ \tilde{c}_1(f^* L_m)$ vanishes on $\Omega_*(X)_{D_*}$.*

Proof. We proceed by induction on $\dim_k Z$. Since the operators $\tilde{c}_1(L)$ are \mathbb{L}_* -linear and commute with each other, it suffices to show that the operator in question vanishes on elements $g : Y \rightarrow X$ of $\mathcal{M}(X)_{D_*}$. The identity

$$\tilde{c}_1(f^* L_1) \circ \dots \circ \tilde{c}_1(f^* L_m)(g : Y \rightarrow X) = g_*((\text{Id}_Y, (fg)^* L_1, \dots, (fg)^* L_m))$$

reduces us to the case $X = Y$, $g = \text{Id}_Y$, that is, it suffices to show that

$$(\text{Id}_Y, f^* L_1, \dots, f^* L_m) = 0 \text{ in } \Omega_*(Y)_{D_*},$$

assuming Id_Y is in $\mathcal{M}(Y)_{D_*}$ and $m > \dim_k Z$.

We may assume that Y is irreducible. As before, we may assume that $|D_i| \neq Y$ for all i . Using the formal group law, we reduce to the case of very ample line bundles L_i (see for example the proof of [6, lemma 3.2.6]).

Assume that $r \geq 1$. Then D_1 is the leading pseudo-divisor for Id_Y and by lemma 1.4, $\text{div} D_1$ is a simple normal crossing divisor on Y .

If $\dim_k Z = 0$, all the line bundles are trivial, hence have a nowhere vanishing section. We may then use the relations in $\langle \mathcal{R}_*^{Sect} \rangle (Y)_{D_*}$ (for the empty divisor) to conclude that $(\text{Id}_Y, f^* L_1, \dots, f^* L_m) = 0$.

Suppose now that $\dim_k Z > 0$. Let s be a section of L_m . By Bertini's theorem, we may choose s so that $s = 0$ is a smooth divisor $\tilde{i} : \tilde{Z} \rightarrow Z$ on Z . Let H be the divisor of $f^* s$ with inclusion $i : |H| \rightarrow Y$, and let $\tilde{f} : |H| \rightarrow \tilde{Z}$ be the induced morphism.

By Bertini's theorem again, we may choose s so that $H + D_1$ is a simple normal crossing divisor on Y . In other words, H is in good position with respect to D_* .

By lemma 1.14, we have the identity in $\Omega_*(Y)_{D_*}$

$$[H \rightarrow Y]_{D_*} = [Y; O_Y(H)]_{D_*} = [Y; f^* L_m]_{D_*}.$$

Thus

$$\begin{aligned} & (\text{Id}_Y, f^* L_1, \dots, f^* L_m) \\ &= \tilde{c}_1(f^* L_1) \circ \dots \circ \tilde{c}_1(f^* L_{m-1})([H \rightarrow Y]_{D_*}) \\ &= i_*(\tilde{c}_1(\bar{f}^*(\bar{i}^* L_1)) \circ \dots \circ \tilde{c}_1(\bar{f}^*(\bar{i}^* L_{m-1}))([H \rightarrow |H|]_{D_*})). \end{aligned}$$

As this last element is zero by our induction hypothesis, the lemma is proved in case $r \geq 1$. If $r = 0$, the same proof works, except that we need only assume that \bar{Z} and H are smooth. \square

2. INTERSECTION WITH A PSEUDO-DIVISOR

2.1. The intersection map. We construct the intersection map and derive some of its basic properties.

Let D, E_1, \dots, E_r be pseudo-divisors on X . Let $f : Y \rightarrow X$ be in $\mathcal{M}(X)_{D, E_*}$ with Y irreducible, and consider a cobordism cycle

$$\eta := (f : Y \rightarrow X, L_1, \dots, L_m) \in \mathcal{Z}_*(X)_{D, E_*}.$$

Let C be a pseudo-divisor on X with C supported in D . In particular, if $f(Y) \not\subset |D|$, then $\text{div } f^* C$ is an effective Cartier divisor with support contained in $|\text{div } f^* D|$.

We define the element $\underline{C}(\eta)_{E_*} \in \mathbb{L}_* \otimes \underline{\mathcal{Z}}_*(|D|)_{E_*}$ as follows: Suppose $f(Y) \subset |D|$. Let $f^D : Y \rightarrow |D|$ be the morphism induced by f ; in this case $\mathcal{Z}_*(Y)_{D, E_*} = \mathcal{Z}_*(Y)_{E_*}$. Let $\eta_Y := (\text{Id}_Y, L_1, \dots, L_m) \in \mathcal{Z}_*(Y)_{E_*}$. We define

$$\underline{C}(\eta)_{E_*} := 1 \otimes f_*^D(\tilde{c}_1(f^* \mathcal{O}_X(C))(\eta_Y)) \in \mathbb{L}_* \otimes \underline{\mathcal{Z}}_{*-1}(|D|)_{E_*}.$$

If $f(Y) \not\subset |D|$, then D is the leading pseudo-divisor for f , hence $\text{div } f^* D$ is a simple normal crossing divisor on Y and thus $\tilde{C} := \text{div } f^* C$ is also a simple normal crossing divisor on Y , with $|\tilde{C}| \subset |f^* D|$. By lemma 1.4, $\text{div } D$ is admissible with respect to E_* . Since each face of \tilde{C} is a face of $\text{div } f^* D$, \tilde{C} is also admissible and the divisor class $[\tilde{C} \rightarrow |f^* D|]_{E_*} \in \mathbb{L}_* \otimes \underline{\mathcal{Z}}_*(|f^* D|)_{E_*}$ is defined. We let $f^D : |f^* D| \rightarrow |D|$ be the restriction of f , L_i^D the restriction of L_i to $|f^* D|$, and define

$$\underline{C}(\eta)_{E_*} := f_*^D(\tilde{c}_1(L_1^D) \circ \dots \circ \tilde{c}_1(L_m^D)([\tilde{C} \rightarrow |f^* D|]_{E_*})) \in \mathbb{L}_* \otimes \underline{\mathcal{Z}}_{*-1}(|D|)_{E_*}.$$

We extend this operation to a homomorphism $\underline{C}(-)_{E_*} : \mathbb{L}_* \otimes \mathcal{Z}_*(X)_{D, E_*} \rightarrow \mathbb{L}_* \otimes \underline{\mathcal{Z}}_{*-1}(|D|)_{E_*}$ by \mathbb{L}_* -linearity.

Definition 2.1. Let D, E_1, \dots, E_r be pseudo-divisors on X and let C be a pseudo-divisor on X supported in D . Let $\text{can} : \mathbb{L}_* \otimes \underline{\mathcal{Z}}_{*-1}(|D|)_{D, E_*} \rightarrow \Omega_*(|D|)_{D, E_*}$ be the canonical surjection. The homomorphism

$$C(-)_{E_*} : \mathbb{L}_* \otimes \mathcal{Z}_*(X)_{D, E_*} \rightarrow \Omega_{*-1}(|D|)_{E_*}$$

is defined to be the composition

$$\mathbb{L}_* \otimes \mathcal{Z}_*(X)_{D,E_*} \xrightarrow{\underline{C}(-)_{E_*}} [\mathbb{L}_* \otimes \underline{\mathcal{Z}}_*(|D|)_{E_*}]_{*-1} \xrightarrow{\text{can}} \Omega_{*-1}(|D|)_{E_*}.$$

We will sometimes drop the subscript E_* , writing $C(-)$ for $C(-)_{E_*}$ and $\underline{C}(-)$ for $\underline{C}(-)_{E_*}$, if the context carries the meaning. We will also often ignore the shift by -1 in the grading.

The next two results follow directly from the definitions:

Lemma 2.2. *Let X be a finite type k -scheme with pseudo-divisors D, E_1, \dots, E_r and let C be a pseudo-divisor on X supported in D . Let $g : X' \rightarrow X$ be a morphism of finite type and let $g_D : |g^*D| \rightarrow |D|$ be the restriction of g .*

(1) *Suppose that g is projective. Let η be in $\mathcal{Z}_*(X')_{D,E_*}$. Then $g_*\eta$ is in $\mathcal{Z}_*(X)_{D,E_*}$, and*

$$g_{D*}(\underline{g^*C}(\eta)_{E_*}) = \underline{C}(g_*\eta)_{E_*}$$

in $\mathbb{L}_ \otimes \underline{\mathcal{Z}}_*(|D|)_{E_*}$.*

(2) *Suppose that g is smooth and quasi-projective. Let η be in $\mathcal{Z}_*(X)_{D,E_*}$. Then $g^*\eta$ is in $\mathcal{Z}_*(X')_{D,E_*}$, g_D is smooth and quasi-projective, and*

$$g_D^*(\underline{C}(\eta)_{E_*}) = \underline{g^*C}(g^*\eta)_{E_*}$$

in $\mathbb{L}_ \otimes \underline{\mathcal{Z}}_*(|g^*D|)_{E_*}$.*

Lemma 2.3. *Let X be a finite type k -scheme, with pseudo-divisors D, E_1, \dots, E_r and let C be a pseudo-divisor supported in D . Let L be a line bundle on X , let L^D be the restriction of L to $|D|$ and take η in $\mathcal{Z}_*(X)_{D,E_*}$. Then*

$$\tilde{c}_1(L^D)(\underline{C}(\eta)_{E_*}) = \underline{C}(\tilde{c}_1(L)(\eta))_{E_*}$$

in $\mathbb{L}_ \otimes \underline{\mathcal{Z}}_*(|D|)_{E_*}$.*

Lemma 2.4. *Let $f : Y \rightarrow X$ a morphism in $\mathcal{M}(X)_{D,E_*}$, let C be a pseudo-divisor on X supported in $|D|$ and let L_1, \dots, L_m be line bundles on Y with $m \geq \dim_k Y$. Then*

$$\underline{C}((Y \rightarrow X, L_1, \dots, L_m))_{E_*} = 0$$

in $\mathbb{L}_ \otimes \underline{\mathcal{Z}}_*(|D|)_{E_*}$.*

Proof. We may suppose Y to be irreducible. Write \tilde{C} for f^*C , and let $f^D : |f^*D| \rightarrow |D|$ be the restriction of f . Using lemmas 2.2 and 2.3, we have

$$\underline{C}((f : Y \rightarrow X, L_1, \dots, L_m))_{E_*} = f_*^D(\tilde{c}_1(L_1) \circ \dots \circ \tilde{c}_1(L_m)(\tilde{C}(1_Y^{D,E_*})_{E_*})).$$

If $f(Y) \subset |D|$, then $\tilde{C}(1_Y^{D,E_*}) = \tilde{c}_1(\mathcal{O}_Y(\tilde{C}))(1_Y^{D,E_*})$, and thus

$$\tilde{c}_1(L_1) \circ \dots \circ \tilde{c}_1(L_m)(\tilde{C}(1_Y^{D,E_*})_{E_*}) = (\text{Id}_Y, L_1, \dots, L_m, \mathcal{O}_Y(\tilde{C})) = 0$$

in $\underline{\mathcal{Z}}_{*-1}(Y)_{E*}$, using the relations $\langle \mathcal{R}_*^{Dim} \rangle(Y)_{E*}$. If $f(Y) \not\subset |D|$, then $\tilde{C}(1_Y^{D,E*})$ is a sum of terms of the form

$$a \cdot \iota_*^J((\tilde{C}^J, M_1, \dots, M_s)),$$

with $a \in \mathbb{L}_*$, $\iota^J : \tilde{C}^J \rightarrow Y$ the inclusion of a face of \tilde{C} , and the M_i line bundles on \tilde{C}^J . Thus $\tilde{c}_1(L_1) \circ \dots \circ \tilde{c}_1(L_m)(\tilde{C}(1_Y^{D,E*}))$ is a sum of terms of the form

$$a \cdot \iota_*^J((\tilde{C}^J, M_1, \dots, M_s, \iota^{J*}L_1, \dots, \iota^{J*}L_m)), \quad a \in \mathbb{L}_*.$$

Since $\dim_k \tilde{C}^J < \dim_k Y$ for each face \tilde{C}^J , the terms

$$(\tilde{C}^J, M_1, \dots, M_s, \iota^{J*}L_1, \dots, \iota^{J*}L_m)$$

vanish in $\underline{\mathcal{Z}}_*(\tilde{C}^J)_{E*}$ (using the relations $\langle \mathcal{R}_*^{Dim} \rangle(\tilde{C}^J)_{E*}$), whence the result. \square

Let $F(u_1, \dots, u_m)$ be a power series with \mathbb{L}_* -coefficients, let L_1, \dots, L_m be line bundles on X , and let $f : Y \rightarrow X$ be in $\mathcal{M}(X)_{D,E*}$. Let F_N denote the truncation of F after total degree N . By lemma 2.4, we have, for all $N \geq \dim_k Y$ and all $n \geq 0$,

$$\underline{\mathcal{C}}(F_N(\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_m))([f]))_{E*} = \underline{\mathcal{C}}(F_{N+n}(\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_m))([f]))_{E*}.$$

Thus, for $\eta \in \mathbb{L}_* \otimes \mathcal{Z}_*(X)_{D,E*}$, we may set

$$\underline{\mathcal{C}}(F(\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_m))(\eta))_{E*} := \lim_{N \rightarrow \infty} \underline{\mathcal{C}}(F_N(\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_m))(\eta))_{E*},$$

as the terms in the limit are eventually constant in N .

With this definition, lemma 2.3 extends to power series in the Chern class operators.

Lemma 2.5. *Let D, E_1, \dots, E_r be pseudo-divisors on X and take η in $\mathbb{L}_* \otimes \mathcal{Z}_*(X)_{D,E*}$. Let $f^D : |f^*D| \rightarrow |D|$ be the restriction of f and let $i : |f^*D| \rightarrow Y$ be the inclusion. Let C be a pseudo-divisor on X supported in D , let $F(u_1, \dots, u_m)$ be a power series with \mathbb{L}_* -coefficients, and let L_1, \dots, L_m be line bundles on X . Then*

$$\begin{aligned} \underline{\mathcal{C}}(f_*(\underline{F}(\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_m))(\eta)))_{E*} \\ = f_*^D(\underline{F}(\tilde{c}_1(i^*L_1), \dots, \tilde{c}_1(i^*L_m))(\underline{f^*C}(\eta)_{E*})) \end{aligned}$$

in $\mathbb{L}_* \otimes \underline{\mathcal{Z}}_*(D)_{E*}$.

Lemma 2.6. *Let D, E_1, \dots, E_r be pseudo-divisors on X , let $f : Y \rightarrow X$ be in $\mathcal{M}(X)_{D,E*}$ with Y irreducible, and let $Z \rightarrow Y$ be a smooth codimension one closed subscheme of Y . We suppose that Z is in good position with respect to D, E_* and that $|f^*D| \neq Y$. Let C be an effective Cartier divisor on Y supported in $|f^*D|$.*

(1) *Suppose that no component of Z is contained in $|C|$ and let $i_Z : Z \rightarrow Y$, $i_{CZ} : |i_Z^*C| \rightarrow |C|$, $i_C : |C| \rightarrow Y$ be the inclusions. Then*

$$i_{CZ*}([i_Z^*C \rightarrow |i_Z^*C|]_{E*}) = \tilde{c}_1(i_C^*O_Y(Z))([C \rightarrow |C|]_{E*})$$

in $\Omega_*(|C|)_{E_*}$.

(2) Suppose that no component of Z is contained in $|f^*D|$, and let $i : |f^*D| \rightarrow Y$ be the inclusion. Then

$$C([Z \rightarrow Y]_{D,E_*})_{E_*} = \tilde{c}_1(i^*O_Y(Z))([C \rightarrow |f^*D|]_{E_*})$$

in $\Omega_*(|f^*D|)_{E_*}$.

Proof. We first check that all the terms in (1) and (2) are defined. By assumption, f is in $\mathcal{M}(X)_{D,E_*}$ and $Y \not\subset |f^*D|$, so D is the leading pseudo-divisor for f with respect to D, E_* and $\text{div } f^*D$ is therefore a simple normal crossing divisor on Y . Since Z is in good position with respect to D, E_* , Z is admissible with respect to D, E_* by remark 1.9(1). The divisor $\text{div } f^*D$ is admissible with respect to E_* by lemma 1.4 and thus C is also admissible with respect to E_* . By lemma 1.10(2) each face of $C + Z$ contained in D is admissible with respect to E_* and thus the simple normal crossing divisor i_Z^*C on Z is admissible with respect to E_* .

Next, we note that (1) implies (2). Indeed, assuming that no component of Z is contained in $|f^*D|$, we have

$$C([Z \rightarrow Y]_{D,E_*})_{E_*} = (i_{DC*} \circ i_{CZ*})([i_Z^*C \rightarrow |i_Z^*C|]_{E_*}),$$

where $i_{DC} : |C| \rightarrow |f^*D|$ is the inclusion. Thus, (2) follows by applying i_{DC*} to the identity in (1). We now prove (1).

Write $C = \sum_{i=1}^m n_i C_i$, with each C_i irreducible. The divisor class $[C \rightarrow |C|]_{E_*}$ is a sum over the faces C^J of C ,

$$[C \rightarrow |C|]_{E_*} = \sum_J \iota_*^J([C^J; F_J^{n_1, \dots, n_m}(L_1^J, \dots, L_m^J)]_{E_*}),$$

where $\iota^J : C^J \rightarrow |C|$ is the inclusion, $L_i = O_W(C_i)$ and L_i^J is the restriction of L_i to C^J .

Since no component of Z is contained in $|C|$, it follows that the intersection $C_Z^J := Z \cap C^J$ is transverse, so C_Z^J is a smooth codimension one closed subscheme of C^J . It follows from lemma 1.10(2) that $C_Z^J \subset C^J$ is in good position with respect to E_* .

Thus, the relations in $\langle \mathcal{R}_*^{Sect} \rangle(|C|)_{E_*}$ imply that

$$\tilde{c}_1(i^*O_Y(Z))([C \rightarrow |C|]_{E_*}) = \sum_J i_*^{ZJ}([C_Z^J; F_J^{n_1, \dots, n_m}(L_1^{ZJ}, \dots, L_m^{ZJ})]_{E_*}),$$

where $i^{ZJ} : C_Z^J \rightarrow |C|$ is the inclusion, and L_i^{ZJ} is the restriction of L_i to C_Z^J . By lemma 1.13

$$[i_Z^*C \rightarrow |i_Z^*C|]_{E_*} = \sum_J \iota_*^{ZJ}([C_Z^J; F_J^{n_1, \dots, n_m}(L_1^{ZJ}, \dots, L_m^{ZJ})]_{E_*}),$$

where $\iota_*^{ZJ} : C_Z^J \rightarrow |i_Z^*C| = |C| \cap Z$ is the inclusion, and since $i_*^{ZJ} = i_{CZ*} \circ \iota_*^{ZJ}$, (1) follows. \square

We will need a strengthening of lemma 2.6.

Lemma 2.7. *Let Y be in \mathbf{Sm}_k and irreducible. Let $i_Z : Z \rightarrow Y$ be a codimension one closed subscheme, smooth over k . Let D, E_1, \dots, E_r be pseudo-divisors on Y such that Id_Y is in $\mathcal{M}(Y)_{D, E_*}$ and Z is in good position with respect to D, E_* . Let C be a pseudo-divisor on Y supported in D . We suppose that $|D| \neq Y$ and let $i_D : |D| \rightarrow Y$ be the inclusion. Then*

$$(2.1) \quad C([Z \rightarrow Y]_{D, E_*})_{E_*} = \tilde{c}_1(i_D^* O_Y(Z))([C \rightarrow |D|]_{E_*})$$

in $\Omega_*(|D|)_{E_*}$.

Proof. The condition that Z is in good position with respect to D, E_* implies by remark 1.9(1) that $i_Z : Z \rightarrow Y$ is in $\mathcal{M}(Y)_{D, E_*}$ and thus $C([Z \rightarrow Y]_{D, E_*})_{E_*}$ is defined. Similarly, the simple normal crossing divisor $\text{div} D$ on Y is admissible with respect to E_* by lemma 1.4, hence $\text{div} C$ is admissible and $[C \rightarrow |D|]_{E_*}$ is also defined.

We first reduce to the case of irreducible Z . Write $Z = \sum_{i=1}^m Z_i$ with Z_i irreducible. Then each Z_i is in good position with respect to D, E_* . By lemma 1.15(2)

$$\tilde{c}_1(i_D^* O_Y(Z_i)) \circ \tilde{c}_1(i_D^* O_Y(Z_j))([C \rightarrow |D|]_{E_*}) = 0$$

for all $i \neq j$ and therefore, using the relations $\langle \mathcal{R}_*^{FGL} \rangle(|D|)_{E_*}$, we have

$$\tilde{c}_1(i_D^* O_Y(Z))([C \rightarrow |D|]_{E_*}) = \sum_{i=1}^r \tilde{c}_1(i_D^* O_Y(Z_i))([C \rightarrow |D|]_{E_*}).$$

As $[Z \rightarrow Y]_{D, E_*} = \sum_{i=1}^m [Z_i \rightarrow Y]_{D, E_*}$, it suffices to handle the case of irreducible Z .

In case Z is not contained in $|D|$ the result follows from lemma 2.6, so we may suppose that $Z \subset |D|$. As Z is then a component of $\text{div} D$, Z is admissible with respect to E_* and with respect to D, E_* .

We first consider the case in which Z is not a component of $|\text{div} C|$. By lemma 1.10(2), the simple normal crossing divisor $i_Z^* \text{div} C$ on Y is in good position with respect to E_* , and thus by lemma 1.14, we have the identity

$$(2.2) \quad [\text{div} i_Z^* C \rightarrow Z]_{E_*} = \tilde{c}_1(i_Z^* O_Y(C))(1_Z^{E_*})$$

in $\Omega_*(Z)_{E_*}$. Let $\iota_Z : Z \rightarrow |D|$ be the inclusion. Applying ι_Z to the identity (2.2) and using Lemma 2.6(1) gives us the identities

$$\iota_{Z*}(\tilde{c}_1(i_Z^* O_Y(C))(1_Z^{E_*})) = \iota_{Z*}([\text{div} i_Z^* C \rightarrow Z]_{E_*}) = \tilde{c}_1(i_D^* O_Y(Z))([C \rightarrow |D|]_{E_*}).$$

As $Z \subset |D|$, we have

$$C([Z \rightarrow Y]_{D, E_*})_{E_*} = \iota_{Z*}(\tilde{c}_1(i_Z^* O_Y(C))(1_Z^{E_*})),$$

which yields the result in this case.

We now assume that Z is a component of $\text{div} C$. Write $\text{div} C = \sum_{i=1}^m n_i C_i$, with the C_i irreducible and with $Z = C_1$. For a face C^J of $\text{div} C$, we have

the following diagram of inclusions

$$\begin{array}{ccccc}
 C^J \cap Z & \xrightarrow{\tau^J} & C^J & & \\
 \eta^J \downarrow & & \swarrow \iota^J & & \downarrow i^J \\
 Z & \xrightarrow{\iota_Z} & |D| & \xrightarrow{i_D} & Y \\
 & \searrow i_Z & & & \\
 & & & &
 \end{array}$$

Since Z is in good position with respect to D, E_* , it follows from lemma 1.10 that the Y -schemes $C^J, C^J \cap Z$ are all in $\mathcal{M}(Y)_{E_*}$, and if C^J is not contained in Z , then $C^J \cap Z \subset C^J$ is in good position with respect to E_* .

Since

$$\begin{aligned}
 n_1 \cdot_F u_1 +_F \dots +_F n_m \cdot_F u_m &= F^{n_1, \dots, n_m}(u_1, \dots, u_m) \\
 &= \sum_J u^J F_J^{n_1, \dots, n_m}(u_1, \dots, u_m),
 \end{aligned}$$

we have

$$\tilde{c}_1(O_Y(C)) = \sum_J \tilde{c}_1(O_Y(C_*))^J F_J(\tilde{c}_1(O_Y(C_1)), \dots, \tilde{c}_1(O_Y(C_m))),$$

where $F_J := F_J^{n_1, \dots, n_m}$ and for $J = (j_1, \dots, j_m) \in \{0, 1\}^m$

$$\tilde{c}_1(O_Y(C_*))^J = \tilde{c}_1(O_Y(C_1))^{j_1} \circ \dots \circ \tilde{c}_1(O_Y(C_m))^{j_m}.$$

Since we therefore have

$$\begin{aligned}
 C([Z \rightarrow Y])_{E_*} &= \tilde{c}_1(i_D^* O_Y(C))([Z \rightarrow |D|]_{E_*}) \\
 &= \sum_J \iota_{Y*}(\tilde{c}_1(O_Y(C_*))^J([Z; F_J(O_Y(C_1), \dots, O_Y(C_m))]_{E_*}))
 \end{aligned}$$

and

$$\begin{aligned}
 \tilde{c}_1(O_Y(Z))([C \rightarrow |D|]_{E_*}) \\
 = \sum_J \iota_*^J(\tilde{c}_1(O_Y(Z))([C^J; F_J(O_Y(C_1), \dots, O_Y(C_m))]_{E_*})),
 \end{aligned}$$

it suffices to prove that

$$\begin{aligned}
 (2.3) \quad \iota_*^J(\tilde{c}_1(O_Y(Z))([C^J; F_J(O_Y(C_1), \dots, O_Y(C_m))]_{E_*})) \\
 = \iota_{Z*}(\tilde{c}_1(O_Y(C_*))^J([Z; F_J(O_Y(C_1), \dots, O_Y(C_m))]_{E_*}))
 \end{aligned}$$

in $\Omega_*(|D|)_{E_*}$, for each index J .

Suppose that $J = (0, j_2, \dots, j_m)$, so no component of C^J is contained in Z . Letting $J_i = (0, 1, \dots, 1, 0, \dots, 0)$, with i 1's, we may assume that $J = J_s$ for some $s \geq 1$. We have the sequence of closed subschemes

$$Z \cap C^{J_s} \subset Z \cap C^{J_{s-1}} \subset \dots \subset Z \cap C^{J_1} \subset Z,$$

with $Z \cap C^{J_i}$ a smooth divisor on $Z \cap C^{J_{i-1}}$, in good position with respect to E_* for each $i = 1, \dots, s$ (lemma 1.10(4)). Applying the relations $\langle \mathcal{R}_*^{Sect}(-) \rangle_{E_*}$ repeatedly, we see that

$$\begin{aligned} \tilde{c}_1(O_Y(C_*))^J ([Z; F_J(O_Y(C_1), \dots, O_Y(C_m))]_{E_*}) \\ = \eta_*^J [Z \cap C^J; F_J(O_Y(C_1), \dots, O_Y(C_m))]_{E_*}. \end{aligned}$$

Applying the same relations to the smooth divisor $Z \cap C^J$ on C^J , which by lemma 1.10(2) is in good position with respect to E_* , we have

$$\begin{aligned} \tilde{c}_1(O_Y(Z)) ([C^J; F_J(O_Y(C_1), \dots, O_Y(C_m))]_{E_*}) \\ = \tau_*^J [Z \cap C^J; F_J(O_Y(C_1), \dots, O_Y(C_m))]_{E_*}. \end{aligned}$$

Since $\iota_*^J \circ \tau_*^J = \iota_{Y*} \circ \eta_*^J$, these two identities yield the equality (2.3) in this case.

In case $J = (1, j_2, \dots, j_m)$, then C^J is contained in $Z = C_1$. Letting $J' = (0, j_2, \dots, j_m)$, we have $C^J = Z \cap C^{J'}$, and no component of $C^{J'}$ is contained in Z . Suppose that $J' \neq (0, \dots, 0)$. As above, we have

$$\begin{aligned} \iota_*^{J'} (\tilde{c}_1(O_Y(Z)) ([C^{J'}; F_J(O_Y(C_1), \dots, O_Y(C_m))]_{E_*})) \\ = \iota_{Z*} (\tilde{c}_1(O_Y(C_*))^{J'} ([Z; F_J(O_Y(C_1), \dots, O_Y(C_m))]_{E_*})) \end{aligned}$$

in $\Omega_*(|D|)_{E_*}$. Also, the smooth divisor $C^J = Z \cap C^{J'}$ on $C^{J'}$ is in good position with respect to E_* , and

$$\begin{aligned} \tilde{c}_1(O_Y(Z)) ([C^{J'}; F_J(O_Y(C_1), \dots, O_Y(C_m))]_{E_*}) \\ = \tau_*^{J'} [C^J; F_J(O_Y(C_1), \dots, O_Y(C_m))]_{E_*} \end{aligned}$$

in $\Omega_*(C^{J'})_{E_*}$. As $\iota^J = \iota^J \circ \tau^{J'}$, this yields

$$\begin{aligned} \iota_*^J (\tilde{c}_1(O_Y(Z)) ([C^J; F_J(O_Y(C_1), \dots)]_{E_*})) \\ = \iota_*^{J'} (\tilde{c}_1(O_Y(Z))^2 ([C^{J'}; F_J(O_Y(C_1), \dots)]_{E_*})) \\ = \tilde{c}_1(O_Y(Z)) (\iota_*^{J'} (\tilde{c}_1(O_Y(Z)) ([C^{J'}; F_J(O_Y(C_1), \dots)]_{E_*}))) \\ = \tilde{c}_1(O_Y(Z)) (\iota_{Y*} (\tilde{c}_1(O_Y(C_*))^{J'} ([Z; F_J(O_Y(C_1), \dots)]_{E_*}))) \\ = \iota_{Z*} ((\tilde{c}_1(O_Y(C_*)))^J ([Z; F_J(O_Y(C_1), \dots)]_{E_*})), \end{aligned}$$

verifying (2.3).

If $J' = (0, \dots, 0)$, then $J = (1, 0, \dots, 0)$, $C^J = C_1 = Z$, $i^J = i_Z$, $\iota^J = \iota_Z$ and $\tilde{c}_1(O_Y(C_*))^J = \tilde{c}_1(O_Y(Z))$. The desired relation (2.3) becomes a simple identity in this case, finishing the proof. \square

3. DESCENT TO $\Omega_*(X)_{D, E_*}$

Let X be a finite type k -scheme with pseudo-divisors D, E_1, \dots, E_r and a pseudo-divisor C supported in D . We proceed in a series of steps to show that intersection with C descends to $C(-)_{E_*} : \Omega_*(X)_{D, E_*} \rightarrow \Omega_{*-1}(|D|)_{E_*}$.

Step 1: The descent to $\underline{\mathcal{Z}}_*(X)_{D,E_*}$. Let $\pi : Y \rightarrow Z$ be a smooth morphism with Z and Y in \mathbf{Sm}_k irreducible, L_1, \dots, L_m line bundles on Z with $m > \dim_k Z$, and $f : Y \rightarrow X$ a morphism in $\mathcal{M}(X)_{D,E_*}$. Using lemmas 2.2 and 2.3, it suffices to show that $(f^*C)(\text{Id}_Y, \pi^*L_1, \dots, \pi^*L_m)_{E_*} = 0$ in $\Omega_*(|f^*D|)_{E_*}$. Changing notation, we may assume that $X = Y$, $f = \text{Id}_Y$, and that either $|D| \neq Y$ and $\text{div} D$ is a simple normal crossing divisor on Y , or $|D| = Y$. We need to show that $C(\text{Id}_Y, \pi^*L_1, \dots, \pi^*L_m)_{E_*} = 0$ in $\Omega_*(|D|)_{E_*}$.

If $Y = |D|$, then

$$\begin{aligned} C(\text{Id}_Y, \pi^*L_1, \dots, \pi^*L_m)_{E_*} &= \tilde{c}_1(O_Y(C))(\text{Id}_Y, \pi^*L_1, \dots, \pi^*L_m) \\ &= (\text{Id}_Y, \pi^*L_1, \dots, \pi^*L_m, O_Y), \end{aligned}$$

which is zero in $\Omega_*(|D|)_{E_*} = \Omega_*(Y)_{E_*}$ by the relations $\langle \mathcal{R}_*^{Dim} \rangle(Y)_{E_*}$.

If $Y \neq |D|$ with inclusion $i : |D| \rightarrow Y$ and $\text{div} D$ is a simple normal crossing divisor on Y , then

$$\begin{aligned} C(\text{Id}_Y, \pi^*L_1, \dots, \pi^*L_m)_{E_*} \\ = \tilde{c}_1((\pi \circ i)^*L_1) \circ \dots \circ \tilde{c}_1((\pi \circ i)^*L_m)([C \rightarrow |D|]_{E_*}). \end{aligned}$$

To see that this class vanishes, apply lemma 1.16 to $\pi \circ i : |D| \rightarrow Z$.

Step 2: The descent to $\underline{\Omega}_*(X)_{D,E_*}$. Let $f : Y \rightarrow X$ be in $\mathcal{M}(X)_{D,E_*}$, let $Z \rightarrow Y$ be a codimension one smooth closed subscheme in good position with respect to D, E_* and let C be a pseudo-divisor on X with support in D . We may suppose that Y is irreducible. As in step 1, we reduce to the case $X = Y$ and $f = \text{Id}_Y$, and it suffices to show that

$$C([Y; O_Y(Z)]_{D,E_*})_{E_*} = C([Z \rightarrow Y]_{D,E_*})_{E_*}.$$

If $Y = |D|$, then $\mathcal{M}(Y)_{D,E_*} = \mathcal{M}(Y)_{E_*}$, $[Z \rightarrow Y]_{D,E_*} = [Z \rightarrow Y]_{E_*}$ and the relations $\langle \mathcal{R}_*^{Sect} \rangle(Y)_{E_*}$ yield

$$\begin{aligned} C([Y; O_Y(Z)]_{D,E_*})_{E_*} &= \tilde{c}_1(O_Y(C)) \circ \tilde{c}_1(O_Y(Z))(1_Y^{E_*}) \\ &= \tilde{c}_1(O_Y(C))([Z \rightarrow Y]_{E_*}) \\ &= C([Z \rightarrow Y]_{D,E_*})_{E_*}. \end{aligned}$$

In case $Y \neq |D|$, then $\text{div} D$ is a simple normal crossing divisor on Y , which by lemma 1.4 is admissible with respect to E_* . The divisor $\text{div} C$ has support contained in $|D|$ and is therefore also admissible with respect to E_* . Let $i : |D| \rightarrow Y$ be the inclusion.

Using lemma 2.7, we have

$$\begin{aligned} C([Y; O_Y(Z)]_{D,E_*})_{E_*} &= \tilde{c}_1(i^*O_Y(Z))(C(1_Y^{D,E_*})_{E_*}) \\ &= \tilde{c}_1(i^*O_Y(Z))([C \rightarrow |D|]_{E_*}) \\ &= C([Z \rightarrow Y]_{D,E_*})_{E_*}, \end{aligned}$$

as desired.

Step 3: The descent to $\Omega_*(X)_{D,E_*}$. Let $f : Y \rightarrow X$ be in $\mathcal{M}(X)_{D,E_*}$ and let L and M be line bundles on Y . As above, we may assume that $Y = X$, $f = \text{Id}_Y$, and it suffices to show that

$$C([Y; F_{\mathbb{L}}(L, M)]_{D,E_*})_{E_*} = C([Y; L \otimes M]_{D,E_*})_{E_*}.$$

Using the relations $\langle \mathcal{R}_*^{FGL} \rangle(|D|)_{E_*}$ and lemma 2.5, we have

$$\begin{aligned} C([Y; F_{\mathbb{L}}(L, M)]_{D,E_*})_{E_*} &= C\left(F_{\mathbb{L}}(\tilde{c}_1(L), \tilde{c}_1(M))(1_Y^{D,E_*})\right)_{E_*} \\ &= F_{\mathbb{L}}(\tilde{c}_1(L), \tilde{c}_1(M))(C(1_Y^{D,E_*})_{E_*}) \\ &= \tilde{c}_1(L \otimes M)(C(1_Y^{D,E_*})_{E_*}) \\ &= C([Y; L \otimes M]_{D,E_*})_{E_*}, \end{aligned}$$

as desired.

4. RELATIONS FOR INTERSECTIONS

4.1. Commutativity. We establish the commutativity of the intersection maps. We begin with some preliminary results.

Lemma 4.1. *Let E_1, \dots, E_r be pseudo-divisors on some $Y \in \mathbf{Sm}_k$. Let D and B be effective Cartier divisors on Y . Suppose that Id_Y is in $\mathcal{M}(Y)_{D,E_*} \cap \mathcal{M}(Y)_{B,E_*}$ and $D + B$ is a simple normal crossing divisor on Y . Let C be an effective Cartier divisor on Y with $|C| \subset |D|$. and let $i : |D| \cap |B| \rightarrow |B|$, $i_B : |B| \rightarrow Y$ be the inclusions. Then B is admissible with respect to D, E_* and E_* , and*

$$i_*(i_B^*C([B \rightarrow |B|]_{D,E_*})_{E_*}) = \tilde{c}_1(i_B^*O_Y(C))([B \rightarrow |B|]_{E_*}).$$

Proof. As B is in good position with respect to D, E_* , it follows from remark 1.9(1) that B is admissible with respect to D, E_* . Since Id_Y is in $\mathcal{M}(Y)_{B,E_*}$, B is admissible with respect to E_* by lemma 1.10(1). Thus the terms in the conclusion are all defined.

We note that C is in good position with respect to B, E_* . Let F be an irreducible component of a face of B . Suppose that $F \not\subset |D|$. We apply lemma 1.10(3) with $E = C$, which tells us that $C \cap F$ is a simple normal crossing divisor on F , in good position with respect to E_* . Thus, letting $i_F : F \rightarrow Y$, $\iota_F : |D| \cap F \rightarrow F$ be the inclusions, we have

$$\iota_{F*} \left(i_F^* C(1_F^{D,E_*})_{E_*} \right) = \iota_{F*} ([C \cap F \rightarrow |D| \cap F]_{E_*}) = \tilde{c}_1(i_F^* O_Y(C))(1_F^{E_*}),$$

the first equality being the definition of the intersection and the second following from lemma 1.14. In case $F \subset |D|$, then $\iota_F = \text{Id}_F$ and

$$\iota_{F*} \left(i_F^* C(1_F^{D,E_*})_{E_*} \right) = i_F^* C(1_F^{D,E_*})_{E_*} = \tilde{c}_1(i_F^* O_Y(C))(1_F^{E_*}).$$

Thus, for each face B^J of B with inclusion $\iota^J : B^J \rightarrow |B|$, we have

$$i_* \left(i_B^* C(\iota_*^J(1_{B^J}^{D,E_*}))_{E_*} \right) = \tilde{c}_1(i_B^* O_Y(C))(\iota_*^J(1_{B^J}^{E_*})).$$

For $B = \sum_{i=1}^m n_i B_i$, the divisor class $[B \rightarrow |B|]_{D, E_*}$ is

$$[B \rightarrow |B|]_{D, E_*} = \sum_J F_J(\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_m))(\iota_*^J(1_{B^J}^{D, E_*})),$$

where $L_j = i_B^* O_Y(B_j)$ and $F_J = F_J^{n_1, \dots, n_m}$. Similarly,

$$[B \rightarrow |B|]_{E_*} = \sum_J F_J(\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_m))(\iota_*^J(1_{B^J}^{E_*})).$$

Thus

$$\begin{aligned} i_*(i_B^* C([B \rightarrow |B|]_{D, E_*})_{E_*}) &= \sum_J F_J(\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_m)) \left(i_*(i_B^* C(\iota_*^J(1_{B^J}^{D, E_*}))_{E_*}) \right) \\ &= \sum_J F_J(\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_m)) \left(\tilde{c}_1(i_B^* O_Y(C))(\iota_*^J(1_{B^J}^{E_*})) \right) \\ &= \tilde{c}_1(i_B^* O_Y(C)) \left(\sum_J F_J(\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_m))(\iota_*^J(1_{B^J}^{E_*})) \right) \\ &= \tilde{c}_1(i_B^* O_Y(C))([B \rightarrow |B|]_{E_*}). \end{aligned}$$

□

Write the universal formal group law as $F_{\mathbb{L}}(u, v) = u + v + uvF_{11}(u, v)$ and let $G_{11}(u, v) = vF_{11}(u, v)$.

Lemma 4.2. *Let W be in \mathbf{Sm}_k with pseudo-divisors E_1, \dots, E_r and simple normal crossing divisor D such that Id_W is in $\mathcal{M}(W)_{D, E_*}$. Let C be an effective Cartier divisor on W , with $|C| \subset |D|$ and suppose $C = C_0 + C_1$, with $C_0 > 0$, $C_1 > 0$, and C_1 smooth.*

(1) *We have the identity in $\Omega_*(|D|)_{E_*}$*

$$\begin{aligned} [C \rightarrow |D|]_{E_*} &= [C_0 \rightarrow |D|]_{E_*} + [C_1 \rightarrow |D|]_{E_*} \\ &\quad + G_{11}(\tilde{c}_1(O_W(C_1)), \tilde{c}_1(O_W(C_0)))([C_1 \rightarrow |D|]_{E_*}). \end{aligned}$$

(2) *Let $f : Y \rightarrow W$ be in $\mathcal{M}(W)_{D, E_*}$. Suppose that Y is irreducible and either $f(Y) \subset |D|$ or f^*C_1 is a smooth divisor on Y . Then*

$$\begin{aligned} (f^*C)(1_Y^{D, E_*})_{E_*} &= (f^*C_0)(1_Y^{D, E_*})_{E_*} + (f^*C_1)(1_Y^{D, E_*})_{E_*} \\ &\quad + G_{11}(\tilde{c}_1(O_Y(f^*C_1)), \tilde{c}_1(O_Y(f^*C_0)))((f^*C_1)(1_Y^{D, E_*})_{E_*}) \end{aligned}$$

in $\Omega_(|f^*D|)_{E_*}$.*

Proof. For (2), suppose first that $|f^*D| = Y$. Then $\mathcal{M}(Y)_{D,E_*} = \mathcal{M}(Y)_{E_*}$ and by definition

$$\begin{aligned} (f^*C)(1_Y^{D,E_*})_{E_*} &= \tilde{c}_1(O_Y(f^*C))(1_Y^{E_*}), \\ (f^*C_i)(1_Y^{D,E_*})_{E_*} &= \tilde{c}_1(O_Y(f^*C_i))(1_Y^{E_*}); \quad i = 0, 1. \end{aligned}$$

Since $O_Y(f^*C) = O_Y(f^*C_0) \otimes O_Y(f^*C_1)$ we have

$$\begin{aligned} \tilde{c}_1(O_Y(f^*C)) &= \tilde{c}_1(O_Y(f^*C_0)) + \tilde{c}_1(O_Y(f^*C_1)) \\ &\quad + G_{11}(\tilde{c}_1(O_Y(f^*C_1)), \tilde{c}_1(O_Y(f^*C_0)) \circ \tilde{c}_1(O_Y(f^*C_1))). \end{aligned}$$

Thus (2) follows by applying this identity to $1_Y^{E_*}$.

If $|f^*D| \neq Y$, then (2) is a consequence of (1). Indeed, in this case, f^*D is a simple normal crossing divisor on Y by lemma 1.4 and

$$\begin{aligned} (f^*C)(1_Y^{D,E_*})_{E_*} &= [f^*C \rightarrow |f^*D|]_{E_*} \\ (f^*C_i)(1_Y^{D,E_*})_{E_*} &= [f^*C_i \rightarrow |f^*D|]_{E_*}, \quad i = 0, 1. \end{aligned}$$

Thus, applying the first assertion to the divisor $f^*C = f^*C_0 + f^*C_1$ on Y , we have

$$\begin{aligned} (f^*C)(1_Y^{D,E_*})_{E_*} &= [f^*C \rightarrow |f^*D|]_{E_*} \\ &= [f^*C_0 \rightarrow |f^*D|]_{E_*} + [f^*C_1 \rightarrow |f^*D|]_{E_*} \\ &\quad + G_{11}(\tilde{c}_1(O_Y(f^*C_1)), \tilde{c}_1(O_Y(f^*C_0)))([f^*C_1 \rightarrow |f^*D|]_{E_*}) \\ &= (f^*C_0)(1_Y^{D,E_*})_{E_*} + (f^*C_1)(1_Y^{D,E_*})_{E_*} \\ &\quad + G_{11}(\tilde{c}_1(O_Y(f^*C_1)), \tilde{c}_1(O_Y(f^*C_0)))(f^*C_1)(1_Y^{D_1,E_*})_{E_*}. \end{aligned}$$

We now prove (1). By lemma 1.4, D is admissible with respect to E_* . Thus C , C_0 and C_1 are all admissible with respect to E_* , so the divisor classes $[C \rightarrow |D|]_{E_*}$, $[C_0 \rightarrow |D|]_{E_*}$ and $[C_1 \rightarrow |D|]_{E_*}$ are all defined.

We first reduce to the case of irreducible C_1 . In general, suppose $C_1 = C'_1 + C''_1$ with $C'_1 > 0$, $C''_1 > 0$; as C_1 is admissible with respect to E_* , so are C'_1 and C''_1 .

Since C'_1 and C''_1 are disjoint, applying the relations $\langle \mathcal{R}_*^{Sect} \rangle(-)_{E_*}$ gives

$$\tilde{c}_1(O_W(C'_1))([C'_1 \rightarrow |D|]_{E_*}) = 0 = \tilde{c}_1(O_W(C'_1))([C''_1 \rightarrow |D|]_{E_*}).$$

Noting that $O_W(C') = O_W(C_0) \otimes O_W(C'_1)$, this gives

$$\begin{aligned} G_{11}(\tilde{c}_1(O_W(C''_1)), \tilde{c}_1(O_W(C')))([C''_1 \rightarrow |D|]_{E_*}) \\ = G_{11}(\tilde{c}_1(O_W(C''_1)), F_{\mathbb{L}}(\tilde{c}_1(O_W(C_0)), \tilde{c}_1(C'_1)))([C''_1 \rightarrow |D|]_{E_*}) \\ = G_{11}(\tilde{c}_1(O_W(C''_1)), \tilde{c}_1(O_W(C_0)))([C''_1 \rightarrow |D|]_{E_*}), \end{aligned}$$

and

$$\begin{aligned} G_{11}(\tilde{c}_1(O_W(C''_1)), \tilde{c}_1(O_W(C'_1)))([C''_1 \rightarrow |D|]_{E_*}) \\ = F_{11}(\tilde{c}_1(O_W(C''_1)), \tilde{c}_1(O_W(C'_1)) \circ \tilde{c}_1(C'_1)([C''_1 \rightarrow |D|]_{E_*})) = 0. \end{aligned}$$

Additionally, the definition of the divisor class tells us that

$$[C_1 \rightarrow |D|]_{E_*} = [C'_1 \rightarrow |D|]_{E_*} + [C''_1 \rightarrow |D|]_{E_*}.$$

We therefore have

$$\tilde{c}_1(O_W(C''_1)) \circ \tilde{c}_1(O_W(C'_1))([C_1 \rightarrow |D|]_{E_*}) = 0,$$

giving the identities

$$\begin{aligned} & G_{11}(\tilde{c}_1(O_W(C_1)), \tilde{c}_1(O_W(C_0)))([C_1 \rightarrow |D|]_{E_*}) \\ &= G_{11}(\tilde{c}_1(O_W(C'_1)) + \tilde{c}_1(O_W(C''_1)), \tilde{c}_1(O_W(C_0)))([C_1 \rightarrow |D|]_{E_*}) \\ &= G_{11}(\tilde{c}_1(O_W(C'_1)), \tilde{c}_1(O_W(C_0)))([C'_1 \rightarrow |D|]_{E_*}) \\ &\quad + G_{11}(\tilde{c}_1(O_W(C''_1)), \tilde{c}_1(O_W(C_0)))([C''_1 \rightarrow |D|]_{E_*}). \end{aligned}$$

With these formulas, one easily shows that (1) for the decompositions $C' = C_0 + C'_1$, $C = C' + C''_1$ and $C_1 = C'_1 + C''_1$ implies (1) for $C = C_0 + C_1$.

We now assume C_1 is irreducible. Write $C = \sum_{i=1}^m n_i C_i$, with each C_i irreducible. For each face C^J of C , let $\iota^J : C^J \rightarrow |D|$ be the inclusion; if $J = (1, j_2, \dots, j_m)$, then $C^J \subset C_1$ and we let $\iota_1^J : C^J \rightarrow C_1$ be the inclusion. We write $\iota^1 : C_1 \rightarrow |D|$ for the inclusion $\iota^{(1,0,\dots,0)}$.

Let F_n denote the n -fold sum in the formal group $(F_{\mathbb{L}}, \mathbb{L}_*)$. We have $F_{(0,j_2,\dots,j_m)}^{0,n_2,\dots,n_m}(u_1, u_2, \dots, u_m) = F_{(j_2,\dots,j_m)}^{n_2,\dots,n_m}(u_2, \dots, u_m)$ and $F_{(j_1,\dots,j_m)}^{0,n_2,\dots,n_m} = 0$ if $j_1 \neq 0$.

The identity $F_{\mathbb{L}}(u_1, F_{n-1}(u_2, \dots, u_n)) = F_n(u_1, \dots, u_n)$ gives us the identity

$$(4.1) \quad \sum_J u^J F_J^{n_1,\dots,n_m}(u_1, \dots, u_m) = u_1 + V + u_1 V F_{11}(u_1, V),$$

where

$$V = F^{n_1-1,\dots,n_m}(u_1, \dots, u_m) = \sum_{J'} u^{J'} F_{J'}^{n_1-1,n_2,\dots,n_m}(u_1, u_2, \dots, u_m).$$

Here the sum $\sum_{J'}$ is over all faces of C_0 (with the convention of using indices $J' = (0, j_2, \dots, j_m) \in \{0, 1\}^m$ in case $n_1 = 1$).

Write $u_1 V F_{11}(u_1, V) = u_1 G_{11}(u_1, V)$ as the sum

$$u_1 V F_{11}(u_1, V) = \sum_K u^K F'_K(u_1, \dots, u_m),$$

where the sum is over all indices $K = (1, k_2, \dots, k_m) \in \{0, 1\}^m$ and we follow our usual convention of requiring that F'_K does not involve u_i if $k_i = 0$. The identity (4.1) yields

$$F_{(1,0,\dots,0)}^{n_1,\dots,n_m} = 1 + F_{(1,0,\dots,0)}^{n_1-1,\dots,n_m} + F'_{(1,0,\dots,0)},$$

and for $K = (1, k_2, \dots, k_m) \neq (1, 0, \dots, 0)$ and for $J' = (0, j_2, \dots, j_m)$

$$\begin{aligned} F_K^{n_1,\dots,n_m} &= F_K^{n_1-1,\dots,n_m} + F'_K \\ F_{J'}^{n_1,\dots,n_m} &= F_{J'}^{n_1-1,\dots,n_m}. \end{aligned}$$

Thus, comparing the definitions of $[C \rightarrow |D|]_{E_*}$, $[C_1 \rightarrow |D|]_{E_*}$ and $[C_0 \rightarrow |D|]_{E_*}$, we see that

$$\begin{aligned} [C \rightarrow |D|]_{E_*} - [C_1 \rightarrow |D|]_{E_*} - [C_0 \rightarrow |D|]_{E_*} \\ = \sum_K' \iota_*^K F'_K(\tilde{c}_1(O(C_1)), \dots, \tilde{c}_1(O(C_m)))(1_{C^K}^{E_*}), \end{aligned}$$

where \sum_K' means the sum over all $K = (1, k_2, \dots, k_n) \in \{0, 1\}^m$. We therefore need to show

$$\begin{aligned} (4.2) \quad \sum_K' \iota_*^K F'_K(\tilde{c}_1(O(C_1)), \dots, \tilde{c}_1(O(C_m)))(1_{C^K}^{E_*}) \\ = G_{11}(\tilde{c}_1(O(C_1)), \tilde{c}_1(O(C_0)))([C_1 \rightarrow |D|]_{E_*}). \end{aligned}$$

We note that C_i is in good position with respect to D, E_* for $i = 0, 1$. Let $K = (1, k_2, \dots, k_m)$, with an index $k_i = 1$, and let $K' = (1, k_2, \dots, k_{i-1}, k_i - 1, k_{i+1}, \dots, k_m)$. By lemma 1.10(2), the smooth divisor $C^K = C^{K'} \cap C_i$ on $C^{K'}$ is in good position with respect to E_* . By repeated applications of the relations $\langle \mathcal{R}_*^{Sect} \rangle(-)_{E_*}$, we have

$$\begin{aligned} (4.3) \quad \iota_{1*}^K (F'_K(\tilde{c}_1(O(C_1)), \dots, \tilde{c}_1(O(C_m)))(1_{C^K}^{E_*})) \\ = (F'_K(\tilde{c}_1(O(C_1)), \dots, \tilde{c}_1(O(C_m))) \circ \tilde{c}_1(O(C_*))^{K-1})(1_{C_1}^{E_*}). \end{aligned}$$

where $K - 1 := (0, k_2, \dots, k_n)$. Note that

$$(4.4) \quad G_{11}(u_1, V) = \sum_K' u^{K-1} F'_K(u_1, \dots, u_m).$$

The relations (4.3), (4.4) together with the identities

$$V(\tilde{c}_1(O(C_1)), \dots, \tilde{c}_1(O(C_m))) = \tilde{c}_1(O(C_0)), \quad \iota_*^K = \iota_*^1 \circ \iota_{1*}^K,$$

imply

$$\begin{aligned} \sum_K' \iota_*^K \left(F'_K(\tilde{c}_1(O(C_1)), \dots, \tilde{c}_1(O(C_m)))(1_{C^K}^{E_*}) \right) \\ = \sum_K' \iota_{1*} \left(\tilde{c}_1(O(C_*))^{K-1} F'_K(\tilde{c}_1(O(C_1)), \dots, \tilde{c}_1(O(C_m)))(1_{C_1}^{E_*}) \right) \\ = \iota_{1*} \left(G_{11}(\tilde{c}_1(O(C_1)), V(\tilde{c}_1(O(C_1)), \dots, \tilde{c}_1(O(C_m))))(1_{C_1}^{E_*}) \right) \\ = G_{11}(\tilde{c}_1(O(C_1)), \tilde{c}_1(O(C_0)))([C_1 \rightarrow |D|]_{E_*}). \end{aligned}$$

This verifies (4.2), completing the proof. \square

Proposition 4.3 (Commutativity). *Let T be in \mathbf{Sm}_k , irreducible, let E_1, \dots, E_r be pseudo-divisors on T , and let D, D' be effective Cartier divisors on T . Suppose that $D + D'$ is a simple normal crossing divisor on T , and that Id_T is in $\mathcal{M}(T)_{D, E_*} \cap \mathcal{M}(T)_{D', E_*}$. Let $i_D : |D| \rightarrow T$ and $i_{D'} : |D'| \rightarrow T$ be the inclusions and let C and C' be effective Cartier divisors on T with*

$|C| \subset |D|$, $|C'| \subset |D'|$. Then C' is admissible with respect to D, E_* , C is admissible with respect to D', E_* and

$$(i_{D'}^* C)([C' \rightarrow |D'|]_{D, E_*})_{E_*} = (i_D^* C')([C \rightarrow |D|]_{D', E_*})_{E_*}$$

in $\Omega_*(|D| \cap |D'|)_{E_*}$.

Proof. Our hypotheses on T , D and D' imply that D is in good position with respect to D', E_* , and D' is in good position with respect to D, E_* . By remark 1.9, C is in good position and admissible with respect to D', E_* , and C' is in good position and admissible with respect to D, E_* , so all the terms in the conclusion are defined.

Write $D = \sum_i n_i D_i$, $D' = \sum_j n'_j D'_j$ with each D_i, D'_j irreducible, and similarly $C = \sum_i m_i D_i$ and $C' = \sum_j m'_j D'_j$ with $0 \leq m_i, 0 \leq m'_j$. We proceed by induction on $m := \sum_i m_i$ and $m' := \sum_j m'_j$.

Suppose $m = m' = 1$; we may suppose $C = D_1$ and $C' = D'_1$. In case $C = C'$, then $C \subset |D'|$ and $C' \subset |D|$, so

$$\begin{aligned} (i_{D'}^* C)([C' \rightarrow |D'|]_{D, E_*})_{E_*} &= \tilde{c}_1(\mathcal{O}(C))([C' \rightarrow |D| \cap |D'|]) \\ &= (i_D^* C')([C \rightarrow |D|]_{D', E_*})_{E_*} \end{aligned}$$

in $\Omega_*(|D| \cap |D'|)_{E_*}$.

Suppose $C \neq C'$. By lemma 1.10(2) the smooth divisor $C \cap C'$ on C' is in good position with respect to E_* ; similarly, the smooth divisor $C \cap C'$ on C is also in good position with respect to E_* . Applying the relations $\langle \mathcal{R}_*^{Sect} \rangle(-)_{E_*}$ gives us the identities

$$\begin{aligned} [C \cap C' \rightarrow |C|]_{E_*} &= \tilde{c}_1(\mathcal{O}_Y(C'))(1_C^{E_*}) \text{ in } \Omega_*(|C|)_{E_*}, \\ [C \cap C' \rightarrow |C'|]_{E_*} &= \tilde{c}_1(\mathcal{O}_Y(C))(1_{C'}^{E_*}) \text{ in } \Omega_*(|C'|)_{E_*}. \end{aligned}$$

Suppose $C \subset |D'|$ but $C' \not\subset |D|$. Then pushing forward the first identity to $|D| \cap |D'|$ gives

$$[C \cap C' \rightarrow |D| \cap |D'|]_{E_*} = i_*^C (\tilde{c}_1(\mathcal{O}_Y(C'))(1_C^{E_*}))$$

in $\Omega_*(|D| \cap |D'|)_{E_*}$, where $i^C : C \rightarrow |D| \cap |D'|$ is the inclusion. This yields the identities in $\Omega_*(|D| \cap |D'|)_{E_*}$

$$\begin{aligned} (i_D^* C')([C \rightarrow |D|]_{D', E_*})_{E_*} &= i_*^C (\tilde{c}_1(\mathcal{O}_Y(C'))(1_C^{E_*})) \\ &= [C \cap C' \rightarrow |D| \cap |D'|]_{E_*} \\ &= (i_{D'}^* C)([C' \rightarrow |D'|]_{D, E_*})_{E_*}. \end{aligned}$$

By symmetry, we have the desired identity if $C' \subset |D|$ but $C \not\subset |D'|$. If $C \cup C' \subset |D| \cap |D'|$, then in $\Omega_*(|D| \cap |D'|)_{E_*}$

$$\begin{aligned} (i_D^* C')([C \rightarrow |D|]_{D', E_*})_{E_*} &= i_*^C (\tilde{c}_1(\mathcal{O}_Y(C'))(1_C^{E_*})) \\ &= [C \cap C' \rightarrow |D| \cap |D'|]_{E_*} \\ &= i_*^{C'} (\tilde{c}_1(\mathcal{O}_Y(C))(1_{C'}^{E_*})) \\ &= (i_{D'}^* C)([C' \rightarrow |D'|]_{D, E_*})_{E_*}. \end{aligned}$$

Finally, if $C \not\subset |D'|$ and $C' \not\subset |D|$, then

$$\begin{aligned} (i_D^* C')([C \rightarrow |D|]_{D', E_*})_{E_*} &= [C \cap C' \rightarrow |D| \cap |D'|]_{E_*} \\ &= (i_{D'}^* C)([C' \rightarrow |D'|]_{D, E_*})_{E_*}. \end{aligned}$$

In the general case, we may assume that D_1 is a component of C . Let $C_1 = D_1$, $C_0 = C - C_1$. By symmetry, it suffices to induct on m and assume the result for the pairs C_0, C' and C_1, C' . Thus

$$\begin{aligned} (i_{D'}^* C_0)([C' \rightarrow |D'|]_{D, E_*})_{E_*} &= (i_D^* C')([C_0 \rightarrow |D|]_{D', E_*})_{E_*} \\ (i_{D'}^* C_1)([C' \rightarrow |D'|]_{D, E_*})_{E_*} &= (i_D^* C')([C_1 \rightarrow |D|]_{D', E_*})_{E_*} \end{aligned}$$

in $\Omega_*(|D| \cap |D'|)_{E_*}$.

The divisor class $[C' \rightarrow |D'|]_{D, E_*}$ is an \mathbb{L}_* -linear combination of cobordism cycles of the form $(g : Y \rightarrow |D'|, M_1, \dots, M_s)$ with $g(Y) \subset |D|$, or with $g^*(C_1)$ a smooth divisor on Y , so we may apply lemma 4.2(2) to give

$$\begin{aligned} (i_{D'}^* C)([C' \rightarrow |D'|]_{D, E_*})_{E_*} &= \\ &= (i_{D'}^* C_0)([C' \rightarrow |D'|]_{D, E_*})_{E_*} + (i_{D'}^* C_1)([C' \rightarrow |D'|]_{D, E_*})_{E_*} \\ &\quad + G_{11}(\tilde{c}_1(O_T(C_1)), \tilde{c}_1(O_T(C_0)))((i_{D'}^* C_1)([C' \rightarrow |D'|]_{D, E_*})_{E_*}). \end{aligned}$$

Using our induction hypothesis, together with lemma 2.5 and lemma 4.2(1), we have the identities in $\Omega_*(|D| \cap |D'|)_{E_*}$:

$$\begin{aligned} (i_{D'}^* C)([C' \rightarrow |D'|]_{D, E_*})_{E_*} &= \\ &= (i_D^* C')([C_0 \rightarrow |D|]_{D', E_*})_{E_*} + (i_D^* C')([C_1 \rightarrow |D|]_{D', E_*})_{E_*} \\ &\quad + G_{11}(\tilde{c}_1(O_T(C_1)), \tilde{c}_1(O_T(C_0)))((i_D^* C')([C_1 \rightarrow |D|]_{D', E_*})_{E_*}) \\ &= (i_D^* C')([C_0 \rightarrow |D|]_{D', E_*} + [C_1 \rightarrow |D|]_{D', E_*})_{E_*} \\ &\quad + G_{11}(\tilde{c}_1(O_T(C_1)), \tilde{c}_1(O_T(C_0)))([C_1 \rightarrow |D|]_{D', E_*})_{E_*} \\ &= (i_D^* C')([C \rightarrow |D|]_{D', E_*})_{E_*}, \end{aligned}$$

as desired. \square

4.2. Linear equivalent pseudo-divisors. We show how to relate the intersection $D_0(-)_D$ with a naive intersection $D_1 \cap (-)$ for linearly equivalent pseudo-divisors D_0, D_1 , in a particular situation.

Proposition 4.4. *Let W be in \mathbf{Sm}_k and let E_1, \dots, E_r be pseudo-divisors on W . Let $f : T \rightarrow W$ be a morphism in \mathbf{Sm}_k and let D_0, D_1 and B be simple normal crossing divisors on T . We suppose that Id_T is in $\mathcal{M}(T)_{D_0, E_*} \cap \mathcal{M}(T)_{B, E_*}$, that $D_0 + B$ is a simple normal crossing divisor on T and that $O_T(D_0) \cong O_T(D_1)$. In addition, suppose that T has an open subscheme V with $|D_1| \subset V$ such that*

- (a) The restriction of f to V is a smooth morphism $f_V : V \rightarrow W$ of relative dimension one and with geometrically irreducible fibers.
- (b) There is a simple normal crossing divisor \bar{B} on W such that \bar{B} is admissible with respect to E_* and such that $B \cap V = f_V^*(\bar{B})$.
- (c) $f_V : V \rightarrow W$ admits a section $s : W \rightarrow V$ and D_1 is the reduced divisor $s(W)$.

Let $i_j : |D_j| \cap |B| \rightarrow |B|$ be the inclusion, $i = 0, 1$. Then the Cartier divisor B on T is admissible with respect to D_0, E_* , the Cartier divisor $B \cap D_1$ on D_1 is admissible with respect to E_* and

$$i_{0*}(D_0([B \rightarrow |B|]_{D_0, E_*})_{E_*}) = i_{1*}([B \cap D_1 \rightarrow |D_1| \cap |B|]_{E_*})$$

in $\Omega_*(|B|)_{E_*}$.

Proof. By lemma 4.1, B is admissible with respect to D_0, E_* and E_* , and

$$i_{0*}(D_0([B \rightarrow |B|]_{D_0, E_*})_{E_*}) = \tilde{c}_1(O_T(D_0))([B \rightarrow |B|]_{E_*})$$

in $\Omega_*(|B|)_{E_*}$.

On the other hand, let B^J be a face of B with non-empty intersection with V . Then by our assumptions on f_V , there is a corresponding face \bar{B}^J of \bar{B} such that $f_V : (B \cap V)^J \rightarrow \bar{B}^J$ is smooth with fibers of dimension one. Let F be an irreducible component of B^J with $F \cap V \neq \emptyset$. Then $f(F)$ is a dense subset of an irreducible component \bar{F} of \bar{B}^J and F is the closure of $f_V^{-1}(\bar{F})$. If E_{i_0} is the leading pseudo-divisor for \bar{F} (with respect to E_*), then E_{i_0} is also the leading pseudo-divisor for F . Since F is admissible with respect to E_* , $f^*E_{i_0} \cap F$ is a simple normal crossing divisor on F . As D_1 is equal to $s(W)$, we see that $(D_1 + \text{div } f^*E_{i_0}) \cap F$ is a simple normal crossing divisor on F , and thus $D_1 \cap F$ is in good position on F with respect to E_* . If \bar{F} has no leading pseudo-divisor for E_* , the same holds for F , and the smooth divisor $D_1 \cap F$ on F is again in good position on F with respect to E_* . If F' is an irreducible component of B^J with $F' \cap V = \emptyset$, then, as $D_1 \cap F' = \emptyset$, $D_1 \cap F'$ is again in good position with respect to E_* . Thus the smooth divisor $D_1 \cap B^J$ on B^J is in good position with respect to E_* .

Thus, by the relations $\langle \mathcal{R}_*^{Sect} \rangle(B^J)_{E_*}$, if B^J is a face of B with non-empty intersection with V , then

$$(4.5) \quad [B^J \cap D_1 \rightarrow |B^J|]_{E_*} = \tilde{c}_1(O_T(D_1))(1_{B^J}^{E_*})$$

in $\Omega_*(|B^J|)_{E_*}$. Similarly, if B^K is a face of B with empty intersection with V , then $B^K \cap D_1 = \emptyset$ and

$$\tilde{c}_1(O_T(D_1))(1_{B^K}^{E_*}) = 0,$$

by applying $\langle \mathcal{R}_*^{Sect} \rangle(B^K)_{E_*}$ in the case of an empty divisor on B^K .

From (b) and (c), the divisor $B \cap D_1$ on D_1 is admissible with respect to E_* . The divisor class $[B \cap D_1 \rightarrow |B| \cap |D_1|]_{E_*}$ is given by a sum of the form

$$[B \cap D_1 \rightarrow |B| \cap |D_1|]_{E_*} = \sum_J F_J(\tilde{c}_1(L_1), \dots)(\bar{c}_*^J(1_{B^J \cap D_1}^{E_*}))$$

where the sum is over the faces B^J of B with $B^J \cap V \neq \emptyset$, and $\bar{\iota}^J : B^J \cap D_1 \rightarrow |B| \cap |D_1|$ is the inclusion. We have a similar description of $[B \rightarrow |B|]_{E_*}$:

$$[B \rightarrow |B|]_{E_*} = \sum_J F_J(\tilde{c}_1(L_1), \dots)(\iota_*^J(1_{B^J}^{E_*})) + \sum_K F_K(\tilde{c}_1(L_1), \dots)(\iota_*^K(1_{B^K}^{E_*}))$$

where the first sum is over faces B^J of B with $B^J \cap V \neq \emptyset$ and the second sum is over faces B^K of B with $B^K \cap V = \emptyset$.

Putting this all together and using (4.5) gives us

$$\begin{aligned} i_{0*}(D_0([B \rightarrow |B|]_{D_0, E_*})_{E_*}) &= \tilde{c}_1(O_T(D_0))([B \rightarrow |B|]_{E_*}) \\ &= \tilde{c}_1(O_T(D_1))([B \rightarrow |B|]_{E_*}) \\ &= \sum_J F_J(\tilde{c}_1(L_1), \dots)\tilde{c}_1(O_T(D_1))(\iota_*^J(1_{B^J}^{E_*})) \\ &= i_{1*}\left(\sum_J F_J(\tilde{c}_1(L_1), \dots)(\bar{\iota}_*^J(1_{B^J \cap D_1}^{E_*}))\right) \\ &= i_{1*}([B \cap D_1 \rightarrow |B| \cap |D_1|]_{E_*}). \end{aligned}$$

□

5. A MOVING LEMMA

Let E_1, \dots, E_r, D be pseudo-divisors on a k -scheme X . We will show in this section that the forgetful map

$$\text{res}_{E_*, D/E_*} : \Omega_*(X)_{E_*, D} \rightarrow \Omega_*(X)_{E_*}$$

is an isomorphism and thus $\text{res}_{E_*/\emptyset} : \Omega_*(X)_{E_*} \rightarrow \Omega_*(X)$ is an isomorphism. This enables us to define the intersection map

$$D(-) : \Omega_*(X) \rightarrow \Omega_{*-1}(X)$$

as the composition

$$\Omega_*(X) \xrightarrow{\text{res}_{D/\emptyset}^{-1}} \Omega_*(X)_D \xrightarrow{D(-)} \Omega_{*-1}(X).$$

5.1. Blow-up diagrams. We discuss some properties of a generalized deformation diagram; this is a reformulation and extension of the discussion in [6, §3.2.1], collecting most of what we will need in the following omnibus construction.

Lemma 5.1. *Let Y be in \mathbf{Sm}_k , let $Z \subset Y$ be a closed subscheme, let $\tau : Y' \rightarrow Y$ be the blowup of Y along Z . Let $Y' := \text{Proj}_{\mathcal{O}_Y} \oplus_{n \geq 0} \mathcal{I}_Z^n$ and let $\mathcal{O}(1) \rightarrow Y'$ be the tautological quotient line bundle, with zero section $s : Y' \rightarrow \mathcal{O}(1)$. Let $\rho : T \rightarrow Y \times \mathbb{A}^1$ be the blowup of $Y \times \mathbb{A}^1$ along $Z \times 0$, let $\langle Y \times 0 \rangle, \langle Z \times \mathbb{A}^1 \rangle \subset T$ be the proper transforms of $Y \times 0, Z \times \mathbb{A}^1$, respectively, and let $T^0 = T \setminus \langle Z \times \mathbb{A}^1 \rangle$. Let $E = \tau^{-1}(Z) \subset Y$ be the exceptional divisor of τ and let $\mathcal{E} = \rho^{-1}(Z \times 0) \subset T$ be the exceptional divisor of ρ . Then*

(1) $\langle Y \times 0 \rangle$ is contained in T^0 .

(2) There is an isomorphism $\alpha : \langle Y \times 0 \rangle \rightarrow Y'$ making the diagram

$$\begin{array}{ccc} \langle Y \times 0 \rangle & \xrightarrow{\alpha} & Y' \\ & \searrow p_1 \circ \rho & \downarrow \tau \\ & & Y \end{array}$$

commute.

(3) There is a morphism of Y -schemes $\pi : T^0 \rightarrow Y'$, such that $\mathcal{E} \cap T^0 = \pi^{-1}(E)$ and with $\alpha = \pi|_{\langle Y \times 0 \rangle}$.

(4) There is an isomorphism of Y' -schemes $\psi : T^0 \rightarrow O(1)$ with $\psi(\langle Y \times 0 \rangle) = s(Y') \subset O(1)$.

Proof. Denote the sheaf of graded \mathcal{O}_Y -algebras $\oplus_{n \geq 0} \mathcal{I}_Z^n$ by \mathcal{B}_\bullet . Let $p : Y \times \mathbb{A}^1 \rightarrow Y$ be the projection. For a sheaf \mathcal{F} on Y we write $\mathcal{F}[t]$ for the sheaf $p^*\mathcal{F}$ on $Y \times \mathbb{A}^1$ and we identify $Y \times \mathbb{A}^1$ with $\text{Spec}_{\mathcal{O}_Y} p_* \mathcal{O}_Y[t]$. Let \mathcal{C}_\bullet be the graded sheaf of $\mathcal{O}_Y[t]$ -algebras $\oplus_{n \geq 0} (\mathcal{I}_Z[t] + (t)\mathcal{O}_Y[t])^n$; by definition, T is the $Y \times \mathbb{A}^1$ -scheme $\text{Proj}_{\mathcal{O}_Y[t]} \mathcal{C}_\bullet$ and $Y' = \text{Proj}_{\mathcal{O}} \mathcal{B}_\bullet$.

The proper transforms $\langle Y \times 0 \rangle$ and $\langle Z \times \mathbb{A}^1 \rangle$ are defined by the homogeneous ideal sheaves in \mathcal{C}_\bullet generated in degree one by (t) and $\mathcal{I}_Z[t]$, respectively. As $\mathcal{I}_Z[t]$ and t generate \mathcal{C}_\bullet^+ as a sheaf of ideals in \mathcal{C}_\bullet , it follows that $\langle Y \times 0 \rangle \cap \langle Z \times \mathbb{A}^1 \rangle = \emptyset$ and thus $\langle Y \times 0 \rangle \subset T^0$, proving (1). Writing $t_{(1)}$ for $t \in \mathcal{C}_1$, the quotient algebra $\mathcal{C}_\bullet/(t_{(1)})$ is $\mathcal{O}_Y[t] \oplus \oplus_{n \geq 1} \mathcal{I}_Z^n$, which is isomorphic to \mathcal{B}_\bullet up to a bounded algebra, hence $p_1 \circ \rho : \langle Y \times \mathbb{A}^1 \rangle \rightarrow Y$ is isomorphic to $\tau : Y' \rightarrow Y$ as a Y -scheme, giving us the isomorphism $\alpha : \langle Y \times \mathbb{A}^1 \rangle \rightarrow Y'$ in (2).

For (3), we have the evident inclusion of sheaves of graded \mathcal{O}_Y -algebras $\mathcal{B}_\bullet \rightarrow \mathcal{C}_\bullet$. Let $Q_\bullet \subset \mathcal{C}_\bullet$ be a sheaf of homogeneous prime ideals. Then $Q_\bullet \cap \mathcal{B}_\bullet \supset \mathcal{B}_\bullet^+$ if and only if $Q_1 \supset \mathcal{I}_Z$ if and only if the corresponding point $[Q_\bullet]$ of T lies in $\langle Z \times \mathbb{A}^1 \rangle$. Thus the inclusion $\mathcal{B}_\bullet \rightarrow \mathcal{C}_\bullet$ induces a well-defined morphism of Y -schemes $\pi : T^0 \rightarrow Y'$. The closed subscheme $E \subset Y'$ is defined by the homogeneous ideal sheaf in \mathcal{B}_\bullet generated by \mathcal{I}_Z (in degree 0), and $\mathcal{E} \subset T$ is similarly defined by the homogeneous ideal sheaf in \mathcal{C}_\bullet generated by $\mathcal{I}_Z[t] + (t)\mathcal{O}_Y[t]$ in degree 0. But if we restrict to a principal open subset U_f of T formed by inverting a section f of $\mathcal{I}_Z \subset \mathcal{C}_1$ over some open subset V of Y , the ideal sheaves on U_f corresponding to $\mathcal{I}_Z \mathcal{C}_\bullet$ and $(\mathcal{I}_Z, t) \mathcal{C}_\bullet$ agree, since $t_{(1)}/f_{(1)}$ is a regular function on U_f and $t = f \cdot (t_{(1)}/f_{(1)})$. Thus $\mathcal{E} \cap T^0$ is defined by the ideal sheaf $\mathcal{I}_Z \mathcal{C}_\bullet$, which shows that $\pi^{-1}(E) = \mathcal{E} \cap T^0$. The composition $\mathcal{B}_\bullet \rightarrow \mathcal{C}_\bullet \rightarrow \mathcal{C}_\bullet/(t_{(1)})$ is the map used to define $\alpha : \langle Y \times \mathbb{A}^1 \rangle \rightarrow Y'$, finishing the proof of (3).

The line bundle $O(1) \rightarrow Y'$ is the affine Y' -scheme $\text{Spec}_{\mathcal{O}_{Y'}} \text{Sym}^* \mathcal{O}(-1)$ and, as $\mathcal{O}(-1)$ is an invertible sheaf, $\text{Sym}^* \mathcal{O}(-1) = \oplus_{m \geq 0} \mathcal{O}(-m)$. The invertible sheaf $\mathcal{O}(-m)$ is the invertible sheaf associated to the graded \mathcal{B}_\bullet -module $\mathcal{B}_\bullet[-m]$, $\mathcal{B}_\bullet[-m]_p = \mathcal{B}_{p-m}$, and so $\text{Sym}^* \mathcal{O}(-1)$ is the sheaf of algebras associated to the polynomial algebra $\mathcal{B}_\bullet[x]$ over \mathcal{B}_\bullet with generator x

in degree 1. The image of the zero section in $O(1)$ is defined by the ideal $(x)\mathcal{B}_\bullet[x] \subset \mathcal{B}_\bullet[x]$.

For the proof of (4), we have the map of graded \mathcal{B}_\bullet -algebras $\psi^* : \mathcal{B}_\bullet[x] \rightarrow \mathcal{C}_\bullet$, which sends $\mathcal{B}_n x^m = \mathcal{I}_Z^n x^m$ to $\mathcal{C}_{m+n} = (\mathcal{I}_Z[t] + t\mathcal{O}_Y[t])^{m+n}$ by setting $\psi^*(ax^m) = at^m$, where a is a section of \mathcal{I}_Z^n .

We claim that ψ^* defines an isomorphism of Y' -schemes $\psi : T^0 \rightarrow O(1)$ and sends $\langle Y \times 0 \rangle$ to $s(Y') \subset O(1)$. To show this, it suffices to handle the case of affine Y , $Y = \text{Spec } A$, and to show in addition that the morphism ψ we define is natural in A .

Let $Z \subset Y$ be defined by an ideal $I \subset A$ and let $B_\bullet = \bigoplus_{n \geq 0} I^n$, $C_\bullet = \bigoplus_{n \geq 0} (I[t] + tA[t])^n$. For $a \in I^m$, we denote the corresponding element of B_m by $a_{(m)}$. For $f \in I$, we have the principal open subschemes $V_f \subset Y'$ defined by $f_{(1)} \in B_1$ and $U_f \subset T$ defined by $f_{(1)} \in C_1$. Y' is covered by the V_f , T^0 is covered by the U_f and $\pi : T^0 \rightarrow Y'$ restricts to $\pi_f : U_f \rightarrow V_f$.

By definition

$$\begin{aligned} V_f &= \text{Spec } B_\bullet[1/f_{(1)}]_0 \\ O(1)|_{V_f} &= \text{Spec } B_\bullet[1/f_{(1)}]_0[x/f_{(1)}] \\ U_f &= \text{Spec } C_\bullet[1/f_{(1)}]_0, \end{aligned}$$

with projection $O(1)|_{V_f} \rightarrow V_f$ given by the inclusion

$$B_\bullet[1/f_{(1)}]_0 \rightarrow B_\bullet[1/f_{(1)}]_0[x/f_{(1)}].$$

The map ψ^* gives rise to the homomorphism of $B_\bullet[1/f_{(1)}]_0$ -algebras

$$\psi_f^* : B_\bullet[1/f_{(1)}]_0[x/f_{(1)}] \rightarrow C_\bullet[1/f_{(1)}]_0$$

with $\psi_f^*(x/f_{(1)}) = t_{(1)}/f_{(1)}$. If g is another element of I , the element $x/f_{(1)}$ maps to $g \cdot [x/(fg)_{(1)}]$ under restriction map for $O(1)|_{V_{fg}} \subset O(1)|_{V_f}$. As $g \cdot [t_{(1)}/(fg)_{(1)}] = t_{(1)}/f_{(1)}$, the maps ψ_f^* and ψ_{fg}^* are compatible with the respective restriction maps for $V_{fg} \subset V_f$ and $U_{fg} \subset U_f$, so the family $(\psi_f^*)_{f \in I}$ gives a well-defined morphism of Y' -schemes $\psi : T^0 \rightarrow O(1)$. Clearly ψ is natural in A . The inverse to ψ_f^* is given by sending $t^i a_{(m)} t_{(1)}^{n-m}/f_{(1)}^n \in (I[t] + tA[t])^n/f_{(1)}^n$, $a \in I^m$, $i \geq 0$, to $f^i \cdot (a_{(m)}/f_{(1)}^m) \cdot (x/f_{(1)})^{n-m+i}$ (use the relation $t = f \cdot (t_{(1)}/f_{(1)})$). Thus ψ is an isomorphism. As $\langle Y \times 0 \rangle \cap U_f$ is defined by the ideal $(t_{(1)}/f_{(1)})$ and the zero section $s(V_f)$ in $O(1)|_{V_f}$ is defined by the ideal $(x/f_{(1)})$, ψ restricts to an isomorphism of $\langle Y \times 0 \rangle$ with $s(Y')$. \square

5.2. Distinguished liftings. Given a finite type k -scheme X with pseudo-divisors E_1, \dots, E_r, D , we describe method for lifting elements of $\mathcal{Z}_*(X)_{E_*}$ to $\Omega_*(X)_{E_*, D}$.

Lemma 5.2. *Let Y be in \mathbf{Sm}_k and let E_1, \dots, E_r, E_{r+1} be effective (non-zero) divisors on Y ; we allow the case $r = 0$, that is, we have only the divisor E_{r+1} . We let E_* denote the sequence E_1, \dots, E_r . Suppose that Id_Y is in $\mathcal{M}(Y)_{E_*}$. Then*

(A) there is a projective birational morphism $\rho : W \rightarrow Y \times \mathbb{P}^1$, with $W \in \mathbf{Sm}_k$, such that, letting \mathcal{E} be the exceptional divisor of ρ , letting $\langle Y \times 0 \rangle$ denote the proper transform to W of $Y \times 0$, and letting $\hat{\tau} : \langle Y \times 0 \rangle \rightarrow Y$ be the restriction of $p_1 \circ \rho$, we have

(5.1)

- (1) The fundamental locus of ρ is contained in $|E_1| \times 0$.
- (2) $\langle Y \times 0 \rangle$ is smooth, the morphism $\hat{\tau} : \langle Y \times 0 \rangle \rightarrow Y$ is birational, with fundamental locus contained in $|E_1| \cap |E_{r+1}|$, and $\hat{\tau}$ is in $\mathcal{M}(Y)_{E_*, E_{r+1}}$.
- (3) The morphism $p_1 \circ \rho : W \rightarrow Y$ is in $\mathcal{M}(Y)_{E_*}$ and $\rho^*(Y \times 0)$ is in good position with respect to E_* .
- (4) The morphism $\rho : W \rightarrow Y \times \mathbb{P}^1$ is in $\mathcal{M}(Y \times \mathbb{P}^1)_{Y \times 0, E_*, E_{r+1}}$.

(B) If $\tau : Y' \rightarrow Y$ is a projective birational morphism with $Y' \in \mathbf{Sm}/k$, with fundamental locus contained in $|E_1| \cap |E_{r+1}|$ and with τ in $\mathcal{M}(Y)_{E_*, E_{r+1}}$, then there is a ρ as above, satisfying (1)-(4), with $\hat{\tau} : \langle Y \times 0 \rangle \rightarrow Y$ isomorphic to the Y -scheme $\tau : Y' \rightarrow Y$, such that

- (5) $\langle Y \times 0 \rangle \cap \langle E_1 \times \mathbb{P}^1 \rangle = \emptyset$,

where $\langle |E_1| \times \mathbb{P}^1 \rangle$ is the proper transform to W of $|E_1| \times \mathbb{P}^1$.

(C) Let $A \subset Y$ be an effective Cartier divisor, in good position with respect to E_* , and suppose that $\tau : Y' \rightarrow Y$ is a morphism satisfying the hypotheses in (B) such that τ^*A is in good position with respect to E_*, E_{r+1} . Then there is a $\rho : W \rightarrow Y \times \mathbb{P}^1$ satisfying (1)-(5) and with $\hat{\tau} : \langle Y \times 0 \rangle \rightarrow Y$ isomorphic to $Y' \rightarrow Y$ as Y -schemes, such that $\rho^*(A \times \mathbb{P}^1)$ is in good position with respect to $Y \times 0, E_*, E_{r+1}$ and $\rho^*(A \times \mathbb{P}^1 + Y \times 0)$ is in good position with respect to E_* .

Proof. We may assume that Y is irreducible. We begin the proof of (A) by showing that there exists a morphism $\tau : Y' \rightarrow Y$ satisfying the hypotheses in (B). We first suppose $r > 0$.

Let $\tilde{\tau} : \tilde{Y} \rightarrow Y$ be the blowup of Y along $E_{(r+1)} := \cap_{i=0}^{r+1} E_i$. As $|E_{(r+1)}| \subset |E_1| \cap |E_{r+1}|$, we see that the fundamental locus F of $\tilde{\tau}$ is contained in $|E_1| \cap |E_{r+1}|$. Let \tilde{E} be the exceptional divisor of $\tilde{\tau}$. Since Id_Y is in $\mathcal{M}(Y)_{E_*}$, it follows that $\tilde{Y} \setminus \tilde{E} \rightarrow Y \setminus F$ is in $\mathcal{M}(Y \setminus F)_{E_*}$, in particular, $\tilde{\tau}^*E_1 \setminus \tilde{E}$ is a simple normal crossing divisor on $\tilde{Y} \setminus \tilde{E}$.

By resolution of singularities, there is a projective birational morphism $\phi : Y' \rightarrow \tilde{Y}$ with $Y' \in \mathbf{Sm}/k$ and with fundamental locus contained in $|\tilde{E}|$, such that $\phi^*(\tilde{\tau}^*E_1)$ is a simple normal crossing divisor on Y' . Letting $\tau : Y' \rightarrow Y$ be the composition $\tilde{\tau} \circ \phi$, it follows immediately that τ has the desired properties. We note that in particular, the exceptional divisor E' of τ is supported in $|\tau^*E_1|$, and τ^*E_1 is a simple normal crossing divisor on Y' .

If $r = 0$, we simply take $\tau : Y' \rightarrow Y$ to be a blowup of Y along a closed subscheme of Y supported in $|E_1|$ so that Y' is in \mathbf{Sm}_k and τ^*E_1 is simple normal crossing divisor on Y' .

Having shown the existence of a projective birational morphism $\tau : Y' \rightarrow Y$ with fundamental locus contained in $|E_1| \cap |E_{r+1}|$, such that τ is in $\mathcal{M}(Y)_{E_*, E_{r+1}}$, we choose any such morphism and proceed with the construction of $\rho : W \rightarrow Y \times \mathbb{P}^1$.

Since τ is a projective birational morphism and Y is smooth, there is a closed subscheme Z_0 with support equal to the fundamental locus of τ (which is contained in $|E_1| \cap |E_{r+1}|$) such that τ is the blow-up of Y along Z_0 . Let Z be the closed subscheme of Y with ideal sheaf $\mathcal{I}_Z = \mathcal{I}_{Z_0} \cdot \mathcal{I}_{E_1}$. Then $|Z| = |E_1|$ and since \mathcal{I}_{E_1} is a locally principal ideal sheaf, τ is also isomorphic to the blowup of Y along Z . Let $q : \mathcal{O}(1) \rightarrow Y'$ be the tautological quotient line bundle on Y' corresponding to this identification.

Let $\rho_1 : W_1 \rightarrow Y \times \mathbb{P}^1$ be the blowup of $Y \times \mathbb{P}^1$ along $Z \times 0$, let $\langle |E_1| \times \mathbb{P}^1 \rangle_1$ and $\langle Y \times 0 \rangle_1$ be the proper transforms of $|E_1| \times \mathbb{P}^1$ and $Y \times 0$ to W_1 , respectively, and let $\mathcal{E}_1 \subset W_1$ be the exceptional divisor of ρ_1 . Let $W_1^0 = \rho_1^{-1}(Y \times \mathbb{A}^1) \cap (W_1 \setminus \langle |E_1| \times \mathbb{P}^1 \rangle_1)$, $F_1 = |\mathcal{E}_1| \cap \langle |E_1| \times \mathbb{P}^1 \rangle_1$ and $U_1 = W_1 \setminus F_1 \supset W_1^0$. By lemma 5.1 we have

- (a) $\langle Y \times 0 \rangle_1 \subset W_1^0$.
- (b) $p_1 \circ \rho_1 : \langle Y \times 0 \rangle_1 \rightarrow Y$ and $\tau : Y' \rightarrow Y$ are isomorphic as Y -schemes.
- (c) There is a Y -morphism $\pi : W_1^0 \rightarrow Y'$ and an isomorphism of Y' -schemes $\psi : W_1^0 \rightarrow \mathcal{O}(1)$, with ψ sending $\langle Y \times 0 \rangle_1$ to the zero section of $\mathcal{O}(1)$ and sending $\mathcal{E}_1 \cap W_1^0$ isomorphically onto $(\tau \circ q)^{-1}(Z)$.

In particular, U_1 is smooth, $\langle Y \times 0 \rangle_1 \subset U_1$, $(\langle Y \times 0 \rangle_1 + \mathcal{E}_1 + \rho_1^*(E_1 \times \mathbb{P}^1)) \cap U_1$ is a simple normal crossing divisor on U_1 and $\rho_1^{-1}(Y \times 0 \cap \bigcap_{i=1}^{r+1} E_i \times \mathbb{P}^1) \cap W_1^0$ is a Cartier divisor on W_1^0 .

We claim that Id_{U_1} is in $\mathcal{M}(U_1)_{E_*} \cap \mathcal{M}(U_1)_{Y \times 0, E_*, E_{r+1}}$. Indeed, let $U'_1 := U_1 \setminus |\rho_1^*(Y \times 0)|$. Then $U'_1 = Y \times (\mathbb{P}^1 \setminus 0)$ as a Y -scheme and $Y \times 0$ pulls back to the empty divisor on U'_1 , so $\text{Id}_{U'_1}$ is in $\mathcal{M}(U'_1)_{E_*} \cap \mathcal{M}(U'_1)_{Y \times 0, E_*, E_{r+1}}$. Similarly, $\text{Id}_{W_1^0}$ is in $\mathcal{M}(W_1^0)_{E_*} \cap \mathcal{M}(W_1^0)_{Y \times 0, E_*, E_{r+1}}$ by properties (a)-(c), and as $U_1 = W_1^0 \cup U'_1$, our claim is verified.

By resolution of singularities, there is a projective birational morphism $\psi : W \rightarrow W_1$ with $W \in \mathbf{Sm}/k$ such that ψ has fundamental locus contained in F_1 , $\psi^*(\mathcal{E}_1 + \langle Y \times 0 \rangle_1 + \rho_1^*(E_1 \times \mathbb{P}^1))$ is a simple normal crossing divisor on W and Id_W is in $\mathcal{M}(W)_{E_*} \cap \mathcal{M}(W)_{Y \times 0, E_*, E_{r+1}}$. Indeed, we may proceed as in the construction of Y' . First blow up the subscheme $\rho_1^{-1}(Y \times 0 \cap \bigcap_{i=1}^{r+1} E_i \times \mathbb{P}^1)$ of W_1 , forming $\psi_1 : W_2 \rightarrow W_1$. Since $\rho_1^{-1}(Y \times 0 \cap \bigcap_{i=1}^{r+1} E_i \times \mathbb{P}^1) \cap U_1$ is a Cartier divisor on U_1 , the fundamental locus of ψ_1 is contained in F_1 . Next, blow up W_2 along a closed subscheme lying over F_1 , $\psi_2 : W \rightarrow W_2$, so that W is in \mathbf{Sm}_k and, letting $\psi := \psi_1 \circ \psi_2$, $\psi^*(\mathcal{E}_1 + \langle Y \times 0 \rangle_1 + \rho_1^*(E_1 \times \mathbb{P}^1))$ a simple normal crossing divisor. Letting $\rho : W \rightarrow Y \times \mathbb{P}^1$ be the composition $\rho_1 \circ \psi$, we claim that ρ satisfies our conditions (1)-(5).

Property (1) is a direct consequence of our choice of Z . Property (2) follows from (b). Properties (3) and (4) are verified in the previous paragraph; to see that $\rho^*(Y \times 0)$ is in good position with respect to E_* , we note that

$\rho^*(Y \times 0) + \rho^*(E_1 \times \mathbb{P}^1) = \langle Y \times 0 \rangle + \mathcal{E} + \rho^*(E_1 \times \mathbb{P}^1)$, which is a simple normal crossing divisor on W . We have constructed W starting with an arbitrary projective birational morphism $\tau : Y' \rightarrow Y$ with $Y' \in \mathbf{Sm}/k$, with fundamental locus contained in $|E_1| \cap |E_{r+1}|$ and with τ in $\mathcal{M}(Y)_{E_*, E_{r+1}}$, and from (b), $\langle Y \times 0 \rangle$ and Y' are isomorphic Y -schemes.

Finally, we see from (a) that the ρ we have constructed satisfies (5). This completes the proof of (A) and (B).

For (C), we note that $\rho_1^*(A \times \mathbb{P}^1) \cap W_1^0 = \pi^*(\tau^*(A))$, and $|\rho_1^*(Y \times 0) \cap W_1^0| = \langle Y \times 0 \rangle \cup |\pi^*(\tau^*E_1)|$. Since $\langle Y \times 0 \rangle$ goes over to the zero-section of $O(1) \rightarrow Y'$, and $\tau^*(A + E_1)$ is a simple normal crossing divisor on Y' , it follows that $\rho_1^*(A \times \mathbb{P}^1 + Y \times 0 + E_1 \times \mathbb{P}^1) \cap W_1^0$ is a simple normal crossing divisor on W_1^0 . Since A is in good position with respect to E_* , we see that $\rho_1^*(A \times \mathbb{P}^1 + E_1 \times \mathbb{P}^1) \setminus |\mathcal{E}_1|$ is a simple normal crossing divisor on $W_1 \setminus |\mathcal{E}_1|$, and thus $\rho_1^*(A \times \mathbb{P}^1 + Y \times 0 + E_1 \times \mathbb{P}^1) \cap U_1$ is a simple normal crossing divisor on U_1 . Taking $\psi : W \rightarrow W_1$ as above, we may blow up W further in a closed subscheme lying over F_1 and change notation so that $\psi^*\rho_1^*(A \times \mathbb{P}^1 + Y \times 0 + E_1 \times \mathbb{P}^1)$ is a simple normal crossing divisor on W , and all the properties of $\rho : W \rightarrow Y \times \mathbb{P}^1$ listed in (A) and (B) still hold. This proves (C). \square

Definition 5.3. Let X be in \mathbf{Sch}_k with pseudo-divisors D_1, \dots, D_n, D .

(1) Let $f : Y \rightarrow X$ be in $\mathcal{M}(X)_{D_*}$ with Y irreducible. Let E_1, \dots, E_r be the sequence of Cartier divisors $\text{div } f^*D_{i_1}, \dots, \text{div } f^*D_{i_r}$ on Y , where $1 \leq i_1 < \dots < i_r \leq n$ and $\{D_{i_1}, \dots, D_{i_r}\}$ is the set of pseudo-divisors D_i such that $f(Y) \not\subset |D_i|$. If $f(Y) \not\subset |D|$, let $E_{r+1} = \text{div } f^*D$, if $f(Y) \subset |D|$, let $E_{r+1} = E_r$; in this latter case, if $r = 0$, we take E_1, \dots, E_r, E_{r+1} to be the empty sequence.

Let $\rho : W \rightarrow Y \times \mathbb{P}^1$ be a birational morphism satisfying the conditions (5.1) for $f : Y \rightarrow X$, E_1, \dots, E_r, E_{r+1} ; in the case of an empty sequence we assume ρ satisfies (5.1) after replacing $|E_1|$ with Y . In either case, the element $\rho^*(Y \times 0)(1_W^{Y \times 0, D, D_*})_{D_*, D} \in \Omega_*(|\rho^*(Y \times 0)|)_{D_*, D}$ is defined. Let $\bar{\rho} : |\rho^*(Y \times 0)| \rightarrow Y$ be the map induced by ρ . We call the element

$$(f \circ \bar{\rho})_*(\rho^*(Y \times 0)(1_W^{Y \times 0, D, D_*})_{D, D_*})$$

of $\Omega_*(X)_{D_*, D}$ a *distinguished lifting* of $f \in \mathcal{M}(X)_{D_*}$.

(2) Let $\eta = (f : Y \rightarrow X, L_1, \dots, L_m)$ be a cobordism cycle on X with $f \in \mathcal{M}(X)_{D_*}$. Choose $\rho : W \rightarrow Y \times \mathbb{P}^1$ as in (1), and let $\tilde{L}_i = (p_1\rho)^*L_i$. We call the element

$$(f \circ \bar{\rho})_*(\tilde{c}_1(\tilde{L}_1) \circ \dots \circ \tilde{c}_1(\tilde{L}_m)(\rho^*(Y \times 0)(1_W^{Y \times 0, D, E_*})_{D, E_*}))$$

of $\Omega_*(X)_{D_*, D}$ a distinguished lifting of η . We extend this notion to arbitrary elements of $\mathbb{L}_* \otimes \mathcal{Z}_*(X)_{D_*}$ by \mathbb{L}_* -linearity.

Remark 5.4. Take $(f : Y \rightarrow X, L_1, \dots, L_r) \in \mathcal{Z}(X)_{D_*}$ and let $\rho : W \rightarrow Y \times \mathbb{P}^1$ be a morphism satisfying the conditions (5.1) for the sequence of

divisors E_1, \dots, E_{r+1} given in definition 5.3. Then the distinguished lifting of $(f : Y \rightarrow X, L_1, \dots, L_r)$ associated to ρ is given by

$$f_*(\tilde{c}_1(L_1) \circ \dots \circ \tilde{c}_1(L_r)((p_1 \circ \rho)_*([\rho^*(Y \times 0) \rightarrow W]_{D_*, D}))),$$

since $\rho^*(Y \times 0)(1_W^{Y \times 0, E_*, E_{r+1}})_{E_*, E_{r+1}} = [\rho^*(Y \times 0) \rightarrow |\rho^*(Y \times 0)|]_{D_*, D}$.

The term “distinguished lifting” is justified by the following result:

Lemma 5.5. *Let D_1, \dots, D_n, D be pseudo-divisors on X , and write $D_* := D_1, \dots, D_n$. Let x be in $\mathbb{L}_* \otimes \mathcal{Z}_*(X)_{D_*}$ and let $x_D \in \Omega_*(X)_{D_*, D}$ be a distinguished lifting. Let $\text{can} : \mathbb{L}_* \otimes \mathcal{Z}_*(X)_{D_*} \rightarrow \Omega_*(X)_{D_*}$ be the canonical map and $\text{res}_{D_*, D/D_*} : \Omega_*(X)_{D_*, D} \rightarrow \Omega_*(X)_{D_*}$ the forgetful map. Then*

$$\text{res}_{D_*, D/D_*}(x_D) = \text{can}(x)$$

in $\Omega_*(X)_{D_*}$.

Proof. It suffices to handle of a cobordism cycle $\eta := (f : Y \rightarrow X, L_1, \dots, L_r)$; as the maps $\mathcal{Z}_*(X)_{D_*} \xrightarrow{\text{can}} \Omega_*(X)_{D_*} \xleftarrow{\text{res}} \Omega_*(X)_{D_*, D}$ are compatible with arbitrary 1st Chern class operators, we reduce to the case $x = (f : Y \rightarrow X)$ with Y irreducible. Let $\rho : W \rightarrow Y \times \mathbb{P}^1$ be the birational morphism used to define x_D and let $i : |\rho^*(Y \times 0)| \rightarrow W$ be the inclusion. The divisors $B_0 := \rho^*(Y \times 0)$ and $B_1 := \rho^*(Y \times 1)$ are both in good position with respect to D_* and are linearly equivalent on W , and Id_W is in $\mathcal{M}(W)_{D_*}$. Hence, by lemma 1.14, we have

$$(5.2) \quad [\rho^*(Y \times 0) \rightarrow W]_{D_*} = [\rho^*(Y \times 1) \rightarrow W]_{D_*}$$

in $\Omega_*(W)_{D_*}$. In addition, Id_W is in $\mathcal{M}(W)_{B_0, D_*}$ and it follows directly from the definition of the operation $(Y \times 0)(-)_{D_*, D}$ that

$$i_*(\rho^*(Y \times 0)(1_W^{Y \times 0, D_*, D})_{D_*, D}) = [\rho^*(Y \times 0) \rightarrow W]_{D_*, D}$$

and thus $x_D = (f \circ p_1)_*([\rho^*(Y \times 0) \rightarrow W]_{D_*, D})$.

Pushing forward the identity (5.2) via $f \circ p_1 \circ \rho$ yields

$$\begin{aligned} \text{res}_{D_*, D/D_*}(x_D) &= \text{res}_{D_*, D/D_*}([\rho^*(Y \times 0) \rightarrow X]_{D_*, D}) \\ &= [\rho^*(Y \times 0) \rightarrow X]_{D_*} \\ &= [\rho^*(Y \times 1) \rightarrow X]_{D_*} \end{aligned}$$

in $\Omega_*(X)_{D_*}$. As $\rho^*(Y \times 1) \rightarrow X$ is isomorphic as an X -scheme to $f : Y \rightarrow X$, the result follows. \square

This yields the following similar result.

Lemma 5.6. *Let η be in $\mathcal{Z}_*(X)_{D_*, D}$. Then*

$$\Phi_{X, D_*, D}(\text{res}_{D_*, D/D_*}(\eta)) = \text{can}(\eta)$$

in $\Omega_*(X)_{D_*, D}$.

Proof. From the relations described in remark 5.8, we reduce to the case of $\eta = 1_X^{D_*,D} \in \mathcal{M}(X)_{D_*,D} \subset \mathcal{M}(X)_{D_*}$, with X irreducible and in \mathbf{Sm}_k . Consider the new sequence of pseudo-divisors D_*, D, D . One sees directly that $\mathcal{M}(X)_{D_*,D} = \mathcal{M}(X)_{D_*,D,D}$, so we have the element $1_X^{D_*,D,D}$ in $\mathcal{M}(X)_{D_*,D,D}$ with $\text{res}_{D_*,D,D/D_*,D}(1_X^{D_*,D,D}) = 1_X^{D_*,D}$. If $\rho : W \rightarrow X \times \mathbb{P}^1$ satisfies the conditions (5.1) for defining the distinguished lifting $\Phi_{X,D_*,D,D}(1_X^{D_*,D,D})$, then the same ρ satisfies the conditions (5.1) for defining the distinguished lifting $\Phi_{X,D_*,D}(1_X^{D_*})$. This implies that

$$\text{res}_{D_*,D,D/D_*,D}(\Phi_{X,D_*,D,D}(1_X^{D_*,D,D})) = \Phi_{X,D_*,D}(\text{res}_{D_*,D/D_*}(1_X^{D_*,D})).$$

By lemma 5.5, we have $\text{res}_{D_*,D,D/D_*,D}(\Phi_{X,D_*,D,D}(1_X^{D_*,D,D})) = \text{can}(1_X^{D_*,D})$ in $\Omega_*(X)_{D_*,D}$, hence $\Phi_{X,D_*,D}(\text{res}_{D_*,D/D_*}(1_X^{D_*,D})) = \text{can}(1_X^{D_*,D})$, as desired. \square

Essential for the construction is the next result.

Proposition 5.7. *Let η be in $\mathcal{Z}_*(X)_{D_*}$, and let $\eta_1, \eta_2 \in \Omega_*(X)_{D_*,D}$ be distinguished liftings of η . Then $\eta_1 = \eta_2$ in $\Omega_*(X)_{D_*,D}$.*

Proof. First of all, we may assume that η is a cobordism cycle ($f : Y \rightarrow X, L_1, \dots, L_r$), with $f \in \mathcal{M}(X)_{D_*}$. Next, it follows from the formula in remark 5.4 that, if τ is a distinguished lifting of (f, L_1, \dots, L_r) , then there is a distinguished lifting $\tilde{1}_Y^{D_*} \in \Omega_*(Y)_{D_*,D}$ of $1_Y^{D_*} \in \Omega_*(Y)_{D_*}$ with

$$\tau = f_*(\tilde{c}_1(L_1) \circ \dots \circ \tilde{c}_1(L_r)(\tilde{1}_Y^{D_*})).$$

Thus, it suffices to consider the case of $X \in \mathbf{Sm}_k$, and to show that two distinguished liftings of $1_X^{D_*} \in \mathcal{M}(X)_{D_*}$ agree in $\Omega_*(X)_{D_*,D}$.

We may assume that X is irreducible. If $|D| = X$, then $\mathcal{Z}_*(X)_{D_*} = \mathcal{Z}_*(X)_{D_*,D}$ and $\Omega_*(X)_{D_*} = \Omega_*(X)_{D_*,D}$. By lemma 5.5, each distinguished lifting of $1_X^{D_*}$ agrees with $1_X^{D_*}$ in $\Omega_*(X)_{D_*}$, hence each two distinguished liftings of $1_X^{D_*}$ agree in $\Omega_*(X)_{D_*,D}$.

Thus, we may assume that D is a Cartier divisor on X . Let η_1 and η_2 be two distinguished liftings of $1_X^{D_*}$. We may also suppose that D_1, \dots, D_n are Cartier divisors on X (we allow the case $n = 0$).

Let $(E_1, \dots, E_r, E_{r+1})$ be the sequence (D_1, \dots, D_n, D) ; in particular, $E_1 = D_1$ for $r \geq 1$, while $E_1 = D$ in case $r = 0$. Suppose for $i = 1, 2$ that η_i is constructed via a birational morphism $\rho_i : W_i \rightarrow X \times \mathbb{P}^1$ satisfying (5.1) for the sequence $(E_1, \dots, E_r, E_{r+1})$. Let Z_i be a subscheme of $X \times \mathbb{P}^1$, supported in $|E_1| \times 0$, such that W_i is the blow-up of $X \times \mathbb{P}^1$ along Z_i , $i = 1, 2$. Let $\bar{\rho}_i : |\rho_i^*(X \times 0)| \rightarrow X$ be the map induced by ρ_i , $i = 1, 2$; thus

$$\eta_i = \bar{\rho}_{i*}(\rho_i^*(X \times 0)(1_{W_i}^{X \times 0, D_*})); \quad i = 1, 2.$$

Let $\phi_1 : T_1 \rightarrow X \times \mathbb{P}^1 \times \mathbb{P}^1$ be the blow-up along $Z_1 \times \mathbb{P}^1$, let $\langle Z_2 \rangle$ denote the proper transform of $p_{13}^*(Z_2)$ to T_1 and let $\psi_1 : T_2 \rightarrow T_1$ be the blow-up of

T_1 along $\langle Z_2 \rangle$, with structure morphism $\phi_2 : T_2 \rightarrow X \times \mathbb{P}^1 \times \mathbb{P}^1$, $\phi_2 = \phi_1 \circ \psi_1$. Let \mathcal{E}_2 be the exceptional divisor of $T_2 \rightarrow X \times \mathbb{P}^1 \times \mathbb{P}^1$.

We claim there is a blow-up $\psi_2 : T \rightarrow T_2$ of T_2 at a closed subscheme Z supported over $|E_1| \times 0 \times 0$, such that T is smooth over k , Id_T is in $\mathcal{M}(T)_{X \times \mathbb{P}^1 \times 0, D_*, D} \cap \mathcal{M}(T)_{X \times 0 \times \mathbb{P}^1, D_*, D}$ and the divisor $X \times \mathbb{P}^1 \times 0 + X \times 0 \times \mathbb{P}^1$ pulls back to a simple normal crossing divisor on T . To see this, define open subschemes U_2 , T_2^1 and T_2^2 of T_2 by

$$\begin{aligned} U_2 &:= T_2 \setminus \phi^{-1}(|E_1| \times 0 \times 0), \\ T_2^1 &:= T_2 \setminus \phi^{-1}(|E_1| \times \mathbb{P}^1 \times 0), \\ T_2^2 &:= T_2 \setminus \phi^{-1}(|E_1| \times 0 \times \mathbb{P}^1). \end{aligned}$$

Let $U_2^1 \subset T_2^1$ and $U_2^2 \subset T_2^2$ be the open subschemes

$$\begin{aligned} U_2^1 &:= T_2 \setminus \phi^{-1}(X \times \mathbb{P}^1 \times 0), \\ U_2^2 &:= T_2 \setminus \phi^{-1}(X \times 0 \times \mathbb{P}^1). \end{aligned}$$

Finally, let $U = (X \setminus |E_1|) \times \mathbb{P}^1 \times \mathbb{P}^1 \subset X \times \mathbb{P}^1 \times \mathbb{P}^1$.

Clearly U_2 is in \mathbf{Sm}_k , $\phi_2 : \phi_2^{-1}(U) \rightarrow U$ is an isomorphism and Id_U is in $\mathcal{M}(U)_{D_*} \cap \mathcal{M}(U)_{X \times 0 \times \mathbb{P}^1, D_*, D} \cap \mathcal{M}(U)_{X \times \mathbb{P}^1 \times 0, D_*, D}$. Next, $U_2^1 = W_1 \times (\mathbb{P}^1 \setminus \{0\})$, so $\text{Id}_{U_2^1}$ is in $\mathcal{M}(U_2^1)_{D_*} \cap \mathcal{M}(U_2^1)_{X \times 0 \times \mathbb{P}^1, D_*, D}$. In addition, $\text{Id}_{U_2^1}$ is in $\mathcal{M}(U_2^1)_{X \times \mathbb{P}^1 \times 0, D_*, D}$, since $X \times \mathbb{P}^1 \times 0$ pulls back to the empty divisor on U_2^1 . As $T_2^1 \setminus U_2^1$ is contained in $\phi_2^{-1}(U)$, this implies that

$$\text{Id}_{T_2^1} \in \mathcal{M}(T_2^1)_{D_*} \cap \mathcal{M}(T_2^1)_{X \times 0 \times \mathbb{P}^1, D_*, D} \cap \mathcal{M}(T_2^1)_{X \times \mathbb{P}^1 \times 0, D_*, D}.$$

By symmetry, we have

$$\text{Id}_{T_2^2} \in \mathcal{M}(T_2^2)_{D_*} \cap \mathcal{M}(T_2^2)_{X \times 0 \times \mathbb{P}^1, D_*, D} \cap \mathcal{M}(T_2^2)_{X \times \mathbb{P}^1 \times 0, D_*, D}.$$

Since $U_2 = T_2^1 \cup T_2^2$, this shows that

$$\text{Id}_{U_2} \in \mathcal{M}(U_2)_{D_*} \cap \mathcal{M}(U_2)_{X \times 0 \times \mathbb{P}^1, D_*, D} \cap \mathcal{M}(U_2)_{X \times \mathbb{P}^1 \times 0, D_*, D}.$$

Thus, there is a blow up $T'_2 \rightarrow T_2$ with fundamental locus contained in $T_2 \setminus U_2$ so that the closed subschemes of $X \times \mathbb{P}^1 \times \mathbb{P}^1$,

$$\begin{aligned} X \times \mathbb{P}^1 \times 0 \cap \cap_{i=1}^s E_i \times \mathbb{P}^1 \times \mathbb{P}^1; \quad s = 1, \dots, n+1, \\ X \times 0 \times \mathbb{P}^1 \cap \cap_{i=1}^s E_i \times \mathbb{P}^1 \times \mathbb{P}^1; \quad s = 1, \dots, n+1, \end{aligned}$$

all pull back to Cartier divisors on T'_2 . Blowing up T'_2 further, again in closed subschemes lying over $T_2 \setminus U_2$, we may resolve the singularities of T'_2 and achieve that $X \times \mathbb{P}^1 \times 0 + X \times 0 \times \mathbb{P}^1$ and $D_1 \times \mathbb{P}^1 \times \mathbb{P}^1$ pull back to simple normal crossing divisors. This gives us the desired blow-up $\psi_2 : T \rightarrow T_2$.

Let $\phi : T \rightarrow X \times \mathbb{P}^1 \times \mathbb{P}^1$ be the composition $\phi_1 \circ \psi_1 \circ \psi_2$. We have the Cartier divisors $\mathcal{D}_0 := \phi^*(X \times 0 \times \mathbb{P}^1)$, $\mathcal{D}_1 := \phi^*(X \times 1 \times \mathbb{P}^1)$, $\mathcal{D}'_0 := \phi^*(X \times \mathbb{P}^1 \times 0)$ and $\mathcal{D}'_1 := \phi^*(X \times \mathbb{P}^1 \times 1)$. We claim that

$$\begin{aligned} \mathcal{D}_0 \text{ is admissible with respect to } \mathcal{D}'_0, D_*, D, \\ \mathcal{D}'_0 \text{ is admissible with respect to } \mathcal{D}_0, D_*, D. \end{aligned}$$

Indeed, by proposition 4.3, \mathcal{D}_0 is admissible with respect to $X \times \mathbb{P}^1 \times 0, D_*, D$. Since $\mathcal{D}'_0 = \phi^*(X \times \mathbb{P}^1 \times 0)$, it follows that \mathcal{D}_0 is admissible with respect to \mathcal{D}'_0, D_*, D . The argument for \mathcal{D}'_0 is the same.

By proposition 4.3 we have

$$(5.3) \quad \mathcal{D}_0([\mathcal{D}'_0 \rightarrow |\mathcal{D}'_0|]_{\mathcal{D}_0, D_*, D})_{D_*, D} = \mathcal{D}'_0([\mathcal{D}_0 \rightarrow |\mathcal{D}_0|]_{\mathcal{D}'_0, D_*, D})_{D_*, D}$$

in $\Omega_*(|\mathcal{D}_0| \cap |\mathcal{D}'_0|)_{D_*, D}$.

We note that T_1 is isomorphic to $W_1 \times \mathbb{P}^1$. Via this isomorphism, let $f : T \rightarrow W_1$ be the composition $p_1 \circ \psi_1 \circ \psi_2$ and let $V = \phi^{-1}(X \times \mathbb{P}^1 \times (\mathbb{P}^1 \setminus \{0\})) \subset T$. As $T \rightarrow T_1$ is an isomorphism over $\phi^{-1}(X \times \mathbb{P}^1 \times (\mathbb{P}^1 \setminus \{0\}))$, we see that the restriction $f_V : V \rightarrow W_1$ of f identifies V with $W_1 \times (\mathbb{P}^1 \setminus \{0\})$. Let $s : W_1 \rightarrow V$ be the 1-section. Letting $\bar{D} = \rho_1^*(X \times 0)$, we have $\mathcal{D}_0 \cap V = f_V^{-1}(\bar{D})$, $\mathcal{D}'_1 \subset V$ and \mathcal{D}'_1 is the reduced divisor $s(W_1)$. As Id_{W_1} is in $\mathcal{M}(W_1)_{X \times 0, D_*, D}$ by construction, the simple normal crossing divisor $\rho_1^*(X \times 0)$ is admissible with respect to D_*, D (lemma 1.10(1)). Finally, $O_T(\mathcal{D}'_0) \cong O_T(\mathcal{D}'_1)$.

Thus, the hypotheses for proposition 4.4 are satisfied (with $E_* = D_*, D$, $B = \mathcal{D}_0$, $D_0 = \mathcal{D}'_0$ and $D_1 = \mathcal{D}'_1$) and we may conclude that $\mathcal{D}_0 \cap \mathcal{D}'_1$ is a simple normal crossing divisor on \mathcal{D}'_1 , admissible with respect to D_*, D , and

$$(5.4) \quad i_{0*}(\mathcal{D}'_0([\mathcal{D}_0 \rightarrow |\mathcal{D}_0|]_{\mathcal{D}'_0, D_*, D})_{D_*, D}) = i_{1*}([\mathcal{D}_0 \cap \mathcal{D}'_1 \rightarrow |\mathcal{D}_0| \cap |\mathcal{D}'_1|]_{D_*, D})$$

in $\Omega_*(|\mathcal{D}_0|)_{D_*, D}$, where $i_0 : |\mathcal{D}_0| \cap |\mathcal{D}'_0| \rightarrow |\mathcal{D}_0|$, $i_1 : |\mathcal{D}_0| \cap |\mathcal{D}'_1| \rightarrow |\mathcal{D}_0|$ are the inclusions. Similarly, $\mathcal{D}'_0 \cap \mathcal{D}_1$ is a simple normal crossing divisor on \mathcal{D}_1 , admissible with respect to D_*, D , and

$$(5.5) \quad i'_{0*}(\mathcal{D}_0([\mathcal{D}'_0 \rightarrow |\mathcal{D}'_0|]_{\mathcal{D}_0, D_*, D})_{D_*, D}) = i'_{1*}([\mathcal{D}'_0 \cap \mathcal{D}_1 \rightarrow |\mathcal{D}'_0| \cap |\mathcal{D}_1|]_{D_*, D})$$

in $\Omega_*(|\mathcal{D}'_0|)_{D_*, D}$, where $i'_0 : |\mathcal{D}'_0| \cap |\mathcal{D}_0| \rightarrow |\mathcal{D}'_0|$, $i'_1 : |\mathcal{D}'_0| \cap |\mathcal{D}_1| \rightarrow |\mathcal{D}'_0|$ are the inclusions.

Putting (5.3), (5.4) and (5.5) together and pushing forward to X gives the identity

$$[\mathcal{D}'_0 \cap \mathcal{D}_1 \rightarrow X]_{D_*, D} = [\mathcal{D}_0 \cap \mathcal{D}'_1 \rightarrow X]_{D_*, D}$$

in $\Omega_*(X)_{D_*, D}$. But via the isomorphisms $\mathcal{D}'_1 \cong W_1$, $\mathcal{D}_1 \cong W_2$, we have

$$\begin{aligned} [\mathcal{D}_0 \cap \mathcal{D}'_1 \rightarrow X]_{D_*, D} &= (\bar{\rho}_1)_*(\rho_1^*(X \times 0)(1_{W_1}^{X \times 0, D_*, D})) = \eta_1, \\ [\mathcal{D}'_0 \cap \mathcal{D}_1 \rightarrow X]_{D_*, D} &= (\bar{\rho}_2)_*(\rho_2^*(X \times 0)(1_{W_2}^{X \times 0, D_*, D})) = \eta_2, \end{aligned}$$

and thus $\eta_1 = \eta_2$, completing the proof. \square

Remark 5.8. Via proposition 5.7, we may speak of *the* distinguished lifting of an element of $\mathbb{L}_* \otimes \mathcal{Z}_*(X)_{D_*}$ to $\Omega_*(X)_{D_*, D}$. We have the following properties of the distinguished lifting:

- (1) Sending $\eta \in \mathbb{L}_* \otimes \mathcal{Z}_*(X)_{D_*}$ to its distinguished lifting $\tilde{\eta}$ defines an \mathbb{L}_* -linear homomorphism $\Phi_{X, D_*, D} : \mathbb{L}_* \otimes \mathcal{Z}_*(X)_{D_*} \rightarrow \Omega_*(X)_{D_*, D}$,

making the diagram

$$\begin{array}{ccc} \mathbb{L}_* \otimes \mathcal{Z}_*(X)_{D_*} & \xrightarrow{\Phi_{X,D_*,D}} & \Omega_*(X)_{D_*,D} \\ & \searrow \text{can} & \downarrow \text{res}_{D_*,D/D_*} \\ & & \Omega_*(X)_{D_*} \end{array}$$

commute.

(2) Given $f : X' \rightarrow X$ projective, we have

$$\Phi_{X,D_*,D} \circ f_* = f_* \circ \Phi_{X',D_*,D}.$$

(3) If L is a line bundle on X , then

$$\Phi_{X,D_*,D} \circ \tilde{c}_1(L) = \tilde{c}_1(L) \circ \Phi_{X,D_*,D}.$$

(4) For $f : X' \rightarrow X$ smooth, we have

$$\Phi_{X',D_*,D} \circ f^* = f^* \circ \Phi_{X,D_*,D}.$$

Property (1) is just lemma 5.5 and proposition 5.7. The properties (2) and (3) follow from the formula in remark 5.4. For (4), suppose $g : Y \rightarrow X$ is in $\mathcal{M}(X)_{D_*}$ and $\rho : W \rightarrow Y \times \mathbb{P}^1$ is used to construct the distinguished lifting of g . As f is smooth, it follows that we may use $W' := X' \times_X W \rightarrow X' \times_X Y \times \mathbb{P}^1$ to construct the distinguished lifting of $f^*(g)$, from which (4) follows easily.

Lemma 5.9. *Let F be in $\mathbb{L}_*[[u_1, \dots, u_m]]$, let $f : W \rightarrow X$ be in $\mathcal{M}(X)_{D_*,D}$, and let L_1, \dots, L_m be line bundles on W . Take an element $\eta \in \mathcal{Z}_*(W)_{D_*}$ and let F_N denote the truncation of F after total degree N . Then*

$$\begin{aligned} \Phi_{W,D_*,D}(f_*(F_N(\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_m))(\eta))) \\ = f_*(F(\tilde{c}_1(L_1), \dots, \tilde{c}_1(L_m))(\Phi_{W,D_*,D}(\eta))) \end{aligned}$$

for all N sufficiently large.

Proof. This follows from remark 5.8, using the relations $\langle \mathcal{R}_*^{Dim} \rangle(X)_{D_*,D}$. \square

5.3. Lifting divisor classes. We need some information on the distinguished lifting of divisor classes before proving the main moving lemma. We fix pseudo-divisors D_1, \dots, D_n, D on X .

Let $f : Y \rightarrow X$ be in $\mathcal{M}(X)_{D_*}$. We suppose that Y is irreducible, $f(Y) \not\subset |D|$ and $f(Y) \not\subset |D_i|$ for $i = 1, \dots, n$. Let $i : S \rightarrow Y$ be a smooth Cartier divisor on Y , in good position with respect to D_* . As in the proof of lemma 5.2, there is a blow-up $\tau : Y' \rightarrow Y$ with fundamental locus contained in $|\text{div } f^*D_1| \cap |\text{div } f^*D|$ such that $f \circ \tau : Y' \rightarrow X$ is in $\mathcal{M}(X)_{D_*,D}$. Blowing up further, we may assume that τ^*S is a simple normal crossing divisor on Y' , and is in good position with respect to D_*, D . Indeed, this is clearly the case after restriction to $Y' \setminus (|(f \circ \tau)^*D_1| \cap |(f \circ \tau)^*D|)$, and we need only blow Y' up in smooth centers lying over $|(f \circ \tau)^*D_1| \cap |(f \circ \tau)^*D|$ so that $\tau^*S + \text{div}(f \circ \tau)^*D_1$ pulls back to a simple normal crossing divisor.

We apply lemma 5.2 with respect to the sequence of divisors

$$(E_1, \dots, E_r, E_{r+1}) = (\operatorname{div} f^* D_1, \dots, \operatorname{div} f^* D_n, \operatorname{div} f^* D)$$

and the blow-up $Y' \rightarrow Y$, forming a projective birational morphism $\rho : W \rightarrow Y \times \mathbb{P}^1$ with $W \in \mathbf{Sm}_k$ satisfying the conditions (1)-(5) of that lemma, and such that the proper transform $\langle Y \times 0 \rangle \subset W$ is isomorphic as Y -scheme to Y' . By part (C) of lemma 5.2, there is such a W so that $\rho^*(S \times \mathbb{P}^1)$ is a simple normal crossing divisor on W , in good position with respect to $Y \times 0, D_*, D$ and with $\rho^*(S \times \mathbb{P}^1 + Y \times 0)$ in good position with respect to D_* .

Let \mathcal{E} denote the exceptional divisor of ρ , $\langle Y \times 0 \rangle$ the proper transform of $Y \times 0$ and $\langle S \times \mathbb{P}^1 \rangle$ the proper transform of $S \times \mathbb{P}^1$. Let $\tilde{S} = \rho^*(S \times \mathbb{P}^1)$, $\tilde{Y} = \rho^*(Y \times 0)$ and let $i_S : |\tilde{S}| \rightarrow W$ be the inclusion. Let $\bar{\rho}_S : |\rho_S^*(S \times 0)| \rightarrow S$ be the map induced by ρ_S , and $\bar{\rho} : |\tilde{Y}| \rightarrow Y$ the map induced by ρ ,

Lemma 5.10. *Let $q : |\tilde{Y}| \cap |\tilde{S}| \rightarrow X$ be the composition of the inclusion into \tilde{Y} with $f \circ \bar{\rho}$. Then*

$$(1) f_* \bar{\rho}_* \left(\tilde{Y} (1_W^{Y \times 0, D_*})_{D_*, D} \right) = \Phi_{X, D_*, D}([f : Y \rightarrow X]_{D_*}).$$

$$(2) q_* \left(i_S^* \tilde{Y} ([\tilde{S} \rightarrow |\tilde{S}|]_{\tilde{Y}, D_*, D})_{D_*, D} \right) = \Phi_{X, D_*, D}(f_*([S \rightarrow Y]_{D_*})).$$

Proof. The assertion (1) follows from the fact that $\rho : W \rightarrow Y \times \mathbb{P}^1$ satisfies the conditions (1)-(4) of (5.1).

For (2), we first claim that

$$(5.6) \quad i_S^* \tilde{Y} ([\tilde{S} \rightarrow |\tilde{S}|]_{\tilde{Y}, D_*, D})_{D_*, D} = i_S^* \tilde{Y} ([\langle S \times \mathbb{P}^1 \rangle \rightarrow |\tilde{S}|]_{\tilde{Y}, D_*, D})_{D_*, D}$$

in $\Omega_*(|\tilde{Y}| \cap |\tilde{S}|)_{D_*, D}$. Indeed, we may write the simple normal crossing divisor \tilde{S} as

$$\tilde{S} = \langle S \times \mathbb{P}^1 \rangle + A,$$

where A is an effective divisor, supported in $|\tilde{S}| \cap |\mathcal{E}|$. Since the exceptional divisor \mathcal{E} is supported in $|\tilde{Y}|$, A is supported in $|\tilde{Y}|$. From the above decomposition of \tilde{S} , we have

$$[\tilde{S} \rightarrow |\tilde{S}|]_{\tilde{Y}, D_*, D} = [\langle S \times \mathbb{P}^1 \rangle \rightarrow |\tilde{S}|]_{\tilde{Y}, D_*, D} + i_*^A \alpha,$$

where α is a class in $\Omega_*(|A|)_{\tilde{Y}, D_*, D}$, and $i^A : |A| \rightarrow |\tilde{S}|$ is the inclusion. Then

$$\begin{aligned} i_S^* \tilde{Y} ([\tilde{S} \rightarrow |\tilde{S}|]_{\tilde{Y}, D_*, D})_{D_*, D} \\ = i_S^* \tilde{Y} ([\langle S \times \mathbb{P}^1 \rangle \rightarrow |\tilde{S}|]_{\tilde{Y}, D_*, D})_{D_*, D} + i_*^A \tilde{c}_1(i^{A*} O_W(\tilde{Y}))(\alpha). \end{aligned}$$

But $\tilde{Y} = \rho^*(Y \times 0)$, hence $O_W(\tilde{Y}) \cong O_W$ in a neighborhood of $|A|$. Thus $\tilde{c}_1(i^{A*} O_W(\tilde{Y}))(\alpha) = 0$, proving our claim.

We are thus reduced to showing that

$$q_* (i_S^* \tilde{Y} ([\langle S \times \mathbb{P}^1 \rangle \rightarrow |\tilde{S}|]_{\tilde{Y}, D_*, D})_{D_*, D}) = \Phi_{X, D_*, D}(f_*([S \rightarrow Y]_{D_*})).$$

We write W_S for $\langle S \times \mathbb{P}^1 \rangle$; the restriction of ρ defines a projective morphism $\rho_S : W_S \rightarrow S \times \mathbb{P}^1$.

The fact that $\rho^*(S \times \mathbb{P}^1)$ is a simple normal crossing divisor on W , in good position with respect to $Y \times 0, D_*, D$ and with respect to D_* , implies that $\langle S \times \mathbb{P}^1 \rangle$ is a smooth Cartier divisor on W , in good position with respect to $Y \times 0, D_*, D$ and with respect to D_* . By remark 1.9, $p_1 \circ \rho_S : W_S \rightarrow S$ is in $\mathcal{M}(S)_{D_*}$ and $\rho_S : W_S \rightarrow S \times \mathbb{P}^1$ is in $\mathcal{M}(S \times \mathbb{P}^1)_{S \times 0, D_*, D}$. Let $\langle S \times 0 \rangle_S$ denote the proper transform of $S \times 0 \subset S \times \mathbb{P}^1$ to W_S and let $S' \subset Y'$ denote the proper transform of S to Y' . As $\langle Y \times 0 \rangle \cong Y'$ as Y -schemes, it follows that $\langle S \times 0 \rangle_S$ is isomorphic to S' as S -schemes. Since $\tau^*(S)$ is a simple normal crossing divisor on Y' , and is in good position with respect to D_*, D , it follows again from remark 1.9 that $S' \rightarrow S$ is in $\mathcal{M}(S)_{D_*, D}$. Since $\rho^*(S \times \mathbb{P}^1 + Y \times 0)$ is in good position with respect to D_* , lemma 1.10(6) tells us that the divisor $\rho_S^*(S \times 0) = \rho^*(Y \times 0) \cap \langle S \times \mathbb{P}^1 \rangle$ on W_S is in good position with respect to D_* . This shows that $\rho_S : W_S \rightarrow S \times \mathbb{P}^1$ satisfies the conditions (1)-(4) of (5.1), except possibly for the requirements about the support of the fundamental loci of $\langle S \times 0 \rangle_S \rightarrow S$ and $W_S \rightarrow S \times \mathbb{P}^1$; these properties follow directly from the corresponding requirements for $\langle Y \times 0 \rangle \rightarrow Y$ and $W \rightarrow Y \times \mathbb{P}^1$. Thus, we may use W_S to compute the distinguished lifting of $f_*([S \rightarrow Y]_{D_*})$, giving

$$\begin{aligned} \Phi_{X, D_*, D}(f_*[S \rightarrow Y]_{D_*}) &= \Phi_{X, D_*, D}((f \circ i)_*(1_S^{D_*})) \\ &= (f \circ i \circ \bar{\rho}_S)_* \left(\rho_S^*(S \times 0)(1_{W_S}^{S \times 0, D_*, D})_{D_*, D} \right) \\ &= (f \circ \bar{\rho})_* \left(\rho^*(Y \times 0)([\langle S \times \mathbb{P}^1 \rangle \rightarrow W]_{Y \times 0, D_*, D})_{D_*, D} \right) \\ &= q_* \left(i_S^* \tilde{Y}([\langle S \times \mathbb{P}^1 \rangle \rightarrow |\tilde{S}|]_{\tilde{Y}, D_*, D})_{D_*, D} \right) \\ &= q_* \left(i_S^* \tilde{Y}([\tilde{S} \rightarrow |\tilde{S}|]_{\tilde{Y}, D_*, D})_{D_*, D} \right), \end{aligned}$$

as desired. \square

5.4. The proof of the moving lemma. We are now ready to prove the main result of this section.

Theorem 5.11. *Let X be a finite type k -scheme, and D_1, \dots, D_n, D pseudo-divisors on X . Then the forgetful map $\text{res}_{D_*, D/D_*} : \Omega_*(X)_{D_*, D} \rightarrow \Omega_*(X)_{D_*}$ is an isomorphism.*

Proof. By lemma 5.5 and lemma 5.6, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{L}_* \otimes \mathcal{Z}_*(X)_{D_*, D} & \xrightarrow{\text{can}} & \Omega_*(X)_{D_*, D} \\ \text{res}_{D_*, D/D_*} \downarrow & \nearrow \Phi_{X, D_*, D} & \downarrow \text{res}_{D_*, D/D_*} \\ \mathbb{L}_* \otimes \mathcal{Z}_*(X)_{D_*} & \xrightarrow{\text{can}} & \Omega_*(X)_{D_*}. \end{array}$$

Since $\text{can} : \mathbb{L}_* \otimes \mathcal{Z}_*(X)_{D_*} \rightarrow \Omega_*(X)_{D_*}$ is surjective, the surjectivity of $\text{res}_{D_*, D/D_*} : \Omega_*(X)_{D_*, D} \rightarrow \Omega_*(X)_{D_*}$ follows. As $\text{can} : \mathbb{L}_* \otimes \mathcal{Z}_*(X)_{D_*, D} \rightarrow \Omega_*(X)_{D_*, D}$ is surjective, so is

$\Phi_{X,D_*,D}$ and the injectivity of $\text{res}_{D_*,D/D_*}$ will follow if we show that $\Phi_{X,D_*,D}$ descends to an \mathbb{L}_* -linear homomorphism $\bar{\Phi}_{X,D_*,D} : \Omega_*(X)_{D_*} \rightarrow \Omega_*(X)_{D_*,D}$.

First, we show that $\Phi := \Phi_{X,D_*,D}$ descends to $\Phi_1 : \mathbb{L}_* \otimes \underline{\mathcal{Z}}_*(X)_{D_*} \rightarrow \Omega_*(X)_{D_*,D}$. As Φ is \mathbb{L}_* -linear and is compatible with 1st Chern class operators, we need only consider an element of the form $\eta := (f : Y \rightarrow X, \pi^* L_1, \dots, \pi^* L_m)$ of $\langle \mathcal{R}_*^{Dim} \rangle(X)_{D_*}$. Here $f : Y \rightarrow X$ is in $\mathcal{M}(X)_{D_*}$, $\pi : Y \rightarrow Z$ is a smooth morphism to some $Z \in \mathbf{Sm}_k$, L_1, \dots, L_m are line bundles on Z , and $m > \dim_k Z$. By lemma 5.9,

$$\Phi(\eta) = f_* \left(\tilde{c}_1(\pi^* L_1) \circ \dots \circ \tilde{c}_1(\pi^* L_m) (\Phi_{Y,D_*,D}(1_Y^{D_*})) \right).$$

The operator $\tilde{c}_1(\pi^* L_1) \circ \dots \circ \tilde{c}_1(\pi^* L_m)$ is zero on $\Omega_*(Y)_{D_*,D}$ by the relations $\langle \mathcal{R}_*^{Sect} \rangle(Y)_{D_*,D}$, so $\Phi(\eta) = 0$, as desired.

Next, we check that Φ_1 descends to $\Phi_2 : \mathbb{L}_* \otimes \underline{\Omega}_*(X)_{D_*} \rightarrow \Omega_*(X)_{D_*,D}$. Since Φ is \mathbb{L}_* -linear, intertwines the operators $\tilde{c}_1(L)$ on $\underline{\mathcal{Z}}_*(X)_{D_*}$ and on $\Omega(X)_{D_*,D}$, and is compatible with pushforward, it suffices to check that Φ_1 vanishes on elements of the form

$$f_*(\tilde{c}_1(O_Y(S))(1_Y^{D_*}) - [i : S \rightarrow Y]_{D_*}),$$

for $f : Y \rightarrow X$ in $\mathcal{M}(X)_{D_*}$, and $i : S \rightarrow Y$ the inclusion of a smooth divisor in good position with respect to D_* .

By lemma 5.9,

$$\tilde{c}_1(O_Y(S))(\Phi_{Y,D_*,D}(1_Y^{D_*})) = \Phi_{Y,D_*,D}(\tilde{c}_1(O_Y(S))(1_Y^{D_*})).$$

On the other hand, let $\rho : W \rightarrow Y \times \mathbb{P}^1$ be as constructed at the beginning of §5.3 for the pair (Y, S) . Retaining the notation from that section, the Cartier divisor $\rho^*(S \times \mathbb{P}^1)$ is in good position with respect to $Y \times 0, D_*, D$ and $[\rho : W \rightarrow Y \times \mathbb{P}^1]$ is in $\mathcal{M}(Y \times \mathbb{P}^1)_{Y \times 0, D_*, D}$, so by lemma 1.14

$$[\rho^*(S \times \mathbb{P}^1) \rightarrow W]_{Y \times 0, D_*, D} = \tilde{c}_1(\mathcal{O}_W(\rho^*(S \times \mathbb{P}^1)))(1_W^{Y \times 0, D_*, D}).$$

Therefore, with $\bar{\rho} : |\rho^*(Y \times 0) \rightarrow Y$ the map induced by ρ , we have

$$\begin{aligned} & \Phi_{Y,D_*,D}([S \rightarrow Y]_{D_*}) \\ &= (\bar{\rho})_*(\rho^*(Y \times 0)([\rho^*(S \times \mathbb{P}^1) \rightarrow W]_{Y \times 0, D_*, D})_{D_*, D}) \\ & \quad \text{by lemma 5.10} \\ &= (\bar{\rho})_*(\rho^*(Y \times 0)(\tilde{c}_1(\mathcal{O}_W(\rho^*(S \times \mathbb{P}^1)))(1_W^{Y \times 0, D_*, D}))) \\ &= \tilde{c}_1(O_Y(S))(\bar{\rho}_*(\rho^*(Y \times 0)(1_W^{Y \times 0, D_*, D}))) \\ &= \tilde{c}_1(O_Y(S))(\Phi_{Y,D_*,D}(1_Y^{D_*})). \end{aligned}$$

Pushing forward to X gives the desired identity.

Finally, we check that Φ_2 descends to $\bar{\Phi} : \Omega_*(X)_{D_*} \rightarrow \Omega_*(X)_{D_*,D}$. As above, it suffices to show that

$$\Phi_2 \left(f_*(F_{\mathbb{L}}(\tilde{c}_1(L), \tilde{c}_1(M))(1_Y^{D_*}) - \tilde{c}_1(L \otimes M)(1_Y^{D_*})) \right) = 0$$

for each $f : Y \rightarrow X$ in $\mathcal{M}(X)_{D_*}$ and each pair of line bundles L, M on Y , where $F_{\mathbb{L}}$ is the universal formal group law. But

$$\begin{aligned} \Phi_{Y, D_*, D}(F_{\mathbb{L}}(\tilde{c}_1(L), \tilde{c}_1(M))(1_Y^{D_*}) - \tilde{c}_1(L \otimes M)(1_Y^{D_*})) \\ = (F_{\mathbb{L}}(\tilde{c}_1(L), \tilde{c}_1(M)) - \tilde{c}_1(L \otimes M))(\Phi_{Y, D_*, D}(1_Y^{D_*})) \\ = 0, \end{aligned}$$

using the relations $\langle \mathcal{R}_*^{FGL} \rangle(Y)_{D_*, D}$ in $\Omega_*(Y)_{D_*, D}$. Pushing forward to X gives the desired identity.

This completes the descent and the proof of the theorem. \square

6. THE INTERSECTION MAP

We are now in a position to define the intersection map and describe its main properties.

Definition 6.1. Let X be in \mathbf{Sch}_k , D a pseudo-divisor on X . The *intersection map*

$$(6.1) \quad D(-) : \Omega_*(X) \rightarrow \Omega_{*-1}(|D|)$$

is defined as the composition

$$\Omega_*(X) \xrightarrow{\text{res}_{D/\emptyset}^{-1}} \Omega_*(X)_D \xrightarrow{D(-)_D} \Omega_{*-1}(|D|);$$

the map $\text{res}_{D/\emptyset} : \Omega_*(X)_D \rightarrow \Omega_*(X)$ being an isomorphism by theorem 5.11.

As an aid to verifying properties of the intersection map, we prove the following technical result.

Lemma 6.2. Take $X \in \mathbf{Sch}_k$ with pseudo-divisors D_1, D_2 . Then the canonical map

$$\mathbb{L}_* \otimes (\mathcal{Z}_*(X)_{D_1, D_2} \cap \mathcal{Z}_*(X)_{D_2, D_1} \cap \mathcal{Z}_*(X)_{D_1 + D_2}) \rightarrow \Omega_*(X)$$

is surjective.

Proof. As the canonical maps $\mathbb{L}_* \otimes \mathcal{Z}_*(X)_{D_1, D_2} \rightarrow \Omega_*(X)_{D_1, D_2}$ and $\mathbb{L}_* \otimes \mathcal{Z}_*(X) \rightarrow \Omega_*(X)$ are surjective, it follows from lemma 5.6 that $\text{res} \circ \text{can} : \mathbb{L}_* \otimes \mathcal{Z}_*(X)_{D_1, D_2} \rightarrow \Omega_*(X)$ is surjective. To prove the result, we need only show that for $f : Y \rightarrow X$ in $\mathcal{M}(X)_{D_1, D_2}$, $\text{res}_{D_1, D_2/\emptyset}(f)$ is in the image of $\mathbb{L}_* \otimes (\mathcal{Z}_*(X)_{D_1, D_2} \cap \mathcal{Z}_*(X)_{D_2, D_1} \cap \mathcal{Z}_*(X)_{D_1 + D_2}) \rightarrow \Omega_*(X)$.

If either $f(Y) \subset |D_1|$ or $f(Y) \subset |D_2|$, then f is already in the intersection $\mathcal{M}(X)_{D_1, D_2} \cap \mathcal{M}(X)_{D_2, D_1} \cap \mathcal{M}(X)_{D_1 + D_2}$, so we may assume that f^*D_i is a Cartier divisor on Y for $i = 1, 2$. Changing notation, we may replace X with Y and D_1, D_2 with effective Cartier divisors on Y , so that Id_Y is in $\mathcal{M}(Y)_{D_1, D_2}$. Thus D_1 is a simple normal crossing divisor on Y and $D_1 \cap D_2$ is a Cartier divisor on Y . If $g : Z \rightarrow Y$ is in $\mathcal{M}(Y)_{D_1, D_2} \cap \mathcal{M}(X)_{D_1 + D_2}$, then automatically g is in $\mathcal{M}(Y)_{D_2, D_1}$, as, in case $g(Z) \not\subset |D_1| \cup |D_2|$, then $g^*(D_1 + D_2)$ is a simple normal crossing divisors on Z , so g^*D_2 is a simple normal crossing divisor, and as $g^*(D_1) \cap g^*(D_2)$ is a Cartier divisor on Z ,

g is in $\mathcal{M}(Y)_{D_2, D_1}$. Thus we need only show that Id_Y is in the image of $\mathbb{L}_* \otimes (\mathcal{Z}_*(Y)_{D_1, D_2} \cap \mathcal{Z}_*(Y)_{D_1 + D_2}) \rightarrow \Omega_*(Y)$.

Let $\tau : Y' \rightarrow Y$ be a blowup of Y , so that Y' is in \mathbf{Sm}_k and $\tau^*(D_1 + D_2)$ is a simple normal crossing divisor on Y' ; this can be accomplished with the blowup of a closed subscheme Z_0 of Y supported in $|D_2|$. As $D_1 \cap D_2$ is a Cartier divisor on Y , the same holds for $\tau^*D_1 \cap \tau^*D_2$, and clearly τ^*D_1 and τ^*D_2 are simple normal crossing divisors on Y' . Thus $Y' \rightarrow Y$ is in $\mathcal{M}(Y)_{D_1, D_2} \cap \mathcal{M}(Y)_{D_1 + D_2}$, τ^*D_2 is in good position on Y' with respect to D_1, D_2 and τ^*D_1 is in good position on Y' with respect to D_2, D_1 .

Let $Z \subset Y$ be the subscheme with ideal sheaf $\mathcal{I}_Z := \mathcal{I}_{Z_0} \cdot \mathcal{I}_{D_1} \cdot \mathcal{I}_{D_2}$ and let $\rho_1 : W_1 \rightarrow Y \times \mathbb{P}^1$ be the blowup of $Y \times \mathbb{P}^1$ along $Z \times 0$. As in the proof of lemma 5.2, we let $\langle Y \times 0 \rangle_1, \langle Z \times \mathbb{P}^1 \rangle_1$ be respective proper transforms of $Y \times 0, Z \times \mathbb{P}^1$, \mathcal{E}_1 the exceptional divisor of ρ_1 , $W_1^0 := W_1 \setminus |\langle Z \times \mathbb{P}^1 \rangle_1|$ and $U_1 := W_1 \setminus (|\mathcal{E}_1| \cap |\langle Z \times \mathbb{P}^1 \rangle_1|)$. Just as in the proof of lemma 5.2, W_1^0 is isomorphic to the line bundle $O(1) \rightarrow Y'$ (as Y -schemes), with the zero section of $O(1)$ going over to $\langle Y \times 0 \rangle_1 \subset W_1^0$ and with $O(1)|_{D_1 \cup D_2}$ going over to $\mathcal{E}_1 \cap W_1^0$. Thus $\langle Y \times 0 \rangle_1$ is isomorphic to Y' ,

$$\rho_1^*(Y \times 0) \cap U_1 = \rho_1^*(Y \times 0) \cap W_1^0 = \mathcal{E}_1 \cap W_1^0 + \langle Y \times 0 \rangle_1$$

is a simple normal crossing divisor on U_1 and U_1 is smooth over k . Furthermore,

$$\rho_1^*(Y \times 0) \cap \rho_1^*p_1^*(D_1 + D_2) \cap U_1 = \rho_1^*(Y \times 0) \cap \rho_1^*p_1^*(D_1 + D_2) \cap W_1^0 = \mathcal{E}_1 \cap W_1^0$$

is a Cartier divisor. Thus Id_{U_1} is in $\mathcal{M}(U_1)_{D_1, D_2} \cap \mathcal{M}(U_1)_{Y \times 0, D_1, D_2} \cap \mathcal{M}(U_1)_{Y \times 0, D_1 + D_2}$. Thus, we may blow up W_1 in a closed subscheme supported in $W_1 \setminus U_1$, $\psi : W \rightarrow W_1$, so that W is smooth over k , $\rho := \rho_1 \circ \psi$ is in

$$\mathcal{M}(Y \times \mathbb{P}^1)_{Y \times 0, D_1, D_2} \cap \mathcal{M}(Y \times \mathbb{P}^1)_{Y \times 0, D_1 + D_2} \cap \mathcal{M}(Y \times \mathbb{P}^1)_{D_1, D_2}$$

and $\rho^*(Y \times 0) + \rho^*(D_1 \times \mathbb{P}^1)$ is a simple normal crossing divisor on W . Thus $\rho^*(Y \times 0)$ is in good position with respect to D_1, D_2 ; clearly $\rho^*(Y \times 1)$ is in good position with respect to D_1, D_2 .

As $\rho^*(Y \times 0)$ is in good position with respect to D_1, D_2 , lemma 1.14 gives the identity

$$[\rho^*(Y \times 0) \rightarrow Y \times \mathbb{P}^1]_{D_1, D_2} = \rho_*(\tilde{c}_1(\rho^*O_{Y \times \mathbb{P}^1}(Y \times 0))(1_W^{D_1, D_2}))$$

in $\Omega_*(Y \times \mathbb{P}^1)_{D_1, D_2}$. As the smooth divisor $\rho^*(Y \times 1)$ on W is in good position with respect to D_1, D_2 , the relations $\langle \mathcal{R}_*^{Sect} \rangle(W)_{D_1, D_2}$ give the identity

$$[\rho^*(Y \times 1) \rightarrow Y \times \mathbb{P}^1]_{D_1, D_2} = \rho_*(\tilde{c}_1(O_W(\rho^*(Y \times 1)))(1_W^{D_1, D_2}))$$

in $\Omega_*(Y \times \mathbb{P}^1)_{D_1, D_2}$. Since $O_W(\rho^*(Y \times 1)) \cong O_W(\rho^*(Y \times 0))$ and the smooth divisor $\rho^*(Y \times 1)$ is as a Y -scheme isomorphic to Y , we may push forward to Y , giving

$$[\rho^*(Y \times 0) \rightarrow Y]_{D_1, D_2} = [\rho^*(Y \times 1) \rightarrow Y]_{D_1, D_2} = 1_Y^{D_1, D_2}$$

in $\Omega_*(Y)_{D_1, D_2} \cong \Omega_*(Y)$.

On the other hand, as ρ is also in $\mathcal{M}(Y)_{Y \times 0, D_1 + D_2}$, and

$$\underline{Y \times 0} \left([\rho : W \rightarrow Y \times \mathbb{P}^1]_{Y \times 0, D_1 + D_2} \right)_{D_1 + D_2} = [\rho^*(Y \times 0) \rightarrow Y \times \mathbb{P}^1]_{D_1 + D_2}$$

in $\mathbb{L}_* \otimes \underline{\mathcal{Z}}_*(Y \times \mathbb{P}^1)_{D_1 + D_2}$. Pushing forward to Y , the two divisor classes $[\rho^*(Y \times 0) \rightarrow Y]_{D_1 + D_2}$ and $[\rho^*(Y \times 0) \rightarrow Y]_{D_1, D_2}$ are equal in $\mathbb{L}_* \otimes \underline{\mathcal{Z}}_*(Y)$, thus $[\rho^*(Y \times 0) \rightarrow Y]_{D_1, D_2}$ is in

$$\mathbb{L}_* \otimes (\underline{\mathcal{Z}}_*(Y)_{D_1 + D_2} \cap \underline{\mathcal{Z}}_*(Y)_{D_1, D_2}) \subset \mathbb{L}_* \otimes \underline{\mathcal{Z}}_*(Y)$$

and Id_Y is in the image of $\mathbb{L}_* \otimes (\mathcal{Z}_*(Y)_{D_1, D_2} \cap \mathcal{Z}_*(Y)_{D_1 + D_2}) \rightarrow \Omega_*(Y)$, as desired. \square

Proposition 6.3. *Let D be a pseudo-divisor on an $X \in \mathbf{Sch}_k$. The intersection map (6.1) has the following properties.*

- (1) *The map $D(-)$ is \mathbb{L}_* -linear.*
- (2) *Let $f : X' \rightarrow X$ be a projective morphism, $f_D : |f^*D| \rightarrow |D|$ the restriction of f . Then $f_* \circ (f^*D)(-) = D(-) \circ f_*$.*
- (3) *Let $f : X' \rightarrow X$ be a smooth quasi-projective morphism, $f_D : |f^*D| \rightarrow |D|$ the restriction of f . Then $f_D^* \circ D(-) = (f^*D)(-) \circ f^*$.*
- (4) *Let L be a line bundle on X . Then $\tilde{c}_1(L) \circ D(-) = D(-) \circ \tilde{c}_1(L)$.*
- (5) *Take $X' \in \mathbf{Sch}_k$, and let $p : X' \times X \rightarrow X$ be the projection. Then for $\eta' \in \Omega_*(X)$, $\eta' \in \Omega_*(X')$, we have*

$$(p^*D)(\eta' \times \eta) = \eta' \times D(\eta)$$

in $\Omega_(X' \times |D|)$.*

- (6) *Let D' be a second pseudo-divisor. Then the two maps*

$$D(-) \circ D'(-), D'(-) \circ D(-) : \Omega_*(X) \rightarrow \Omega_{*-2}(|D| \cap |D'|)$$

are equal.

- (7) *Let D' be a second pseudo-divisor. Suppose that $O_X(D) \cong O_X(D')$ and let $i : |D| \rightarrow X$, $i' : |D'| \rightarrow X$ be the inclusions. Then*

$$i_* \circ D(-) = i'_* \circ D'(-) : \Omega_*(X) \rightarrow \Omega_{*-1}(X).$$

Proof. The properties (1)-(4) follow directly from lemma 2.2, lemma 2.3 and the definition of the map $D(-)$. For (5), we use (1)-(4) to reduce to the case $X, X' \in \mathbf{Sm}_k$, $\eta = \text{Id}_X$, $\eta' = \text{Id}_{X'}$; in particular, $p : X' \times X \rightarrow X$ is smooth and quasi-projective. As $1_{X'} \times 1_X = 1_{X' \times X} = p^*(1_X)$, we have

$$p^*(D)(1_{X'} \times 1_X) = p^*(D)(p^*(1_X)) = p_D^*(D(1_X)) = 1_{X'} \times D(1_X),$$

the second equality following from (3).

For (6), we use lemma 6.2 to reduce to showing that

$$D(D'(\eta)) = D'(D(\eta))$$

for $\eta \in \mathbb{L}_* \otimes (\mathcal{Z}_*(X)_{D, D'} \cap \mathcal{Z}_*(X)_{D', D} \cap \mathcal{Z}_*(X)_{D + D'})$. Since the maps $D(-), D'(-)$ are \mathbb{L}_* -linear and compatible with 1st Chern class operators, we reduce to showing $D(D'([f : Y \rightarrow X])) = D'(D([f : Y \rightarrow X]))$ for $f \in \mathcal{M}_*(X)_{D, D'} \cap \mathcal{M}_*(X)_{D', D} \cap \mathcal{M}_*(X)_{D + D'}$. As the intersection maps are

compatible with f_* , we reduce to the case $X = Y$ and $f = \text{Id}_Y$. We may assume that Y is irreducible. Since Id_Y is in $\mathcal{M}_*(X)_{D,D'} \cap \mathcal{M}_*(X)_{D',D}$, it is not necessary to apply the inverse of any of the forgetful maps to compute the intersection maps.

If $|D| = |D'| = Y$, then

$$\begin{aligned} D(D'(1_Y^{D,D'})) &= \tilde{c}_1(O_Y(D))(\tilde{c}_1(O_Y(D'))(1_Y)) \\ &= \tilde{c}_1(O_Y(D'))(\tilde{c}_1(O_Y(D))(1_Y)) = D'(D(1_Y^{D',D})). \end{aligned}$$

If $|D| = Y$, but $|D'| \neq Y$, then

$$\begin{aligned} D(D'(1_Y^{D,D'})) &= \tilde{c}_1(O_Y(D))(D'(1_Y^{D,D'})) \\ &= D'(\tilde{c}_1(O_Y(D))(1_Y^{D'})) = D'(D(1_Y^{D,D'})). \end{aligned}$$

By symmetry, we have the desired identity in case $|D'| = Y$, but $|D| \neq Y$ as well.

Suppose both D and D' are Cartier divisors on Y . Since Id_Y is in $\mathcal{M}_*(Y)_{D+D'} \cap \mathcal{M}(Y)_{D,D'} \cap \mathcal{M}(Y)_{D',D}$, $D + D'$ is simple normal crossing divisor. We may then apply proposition 4.3 to yield

$$i_{D'}^*(D)([D' \rightarrow |D'|]_D) = i_D^*(D')([D \rightarrow |D|]_{D'})$$

in $\Omega_*(Y)$. Thus

$$\begin{aligned} D(D'(1_Y^{D,D'})) &= i_{D'}^*(D)(D'(1_Y^{D,D'})) \\ &= i_{D'}^*(D)([D' \rightarrow |D'|]_D) \\ &= i_D^*(D')([D \rightarrow |D|]_{D'}) \\ &= D'(D(1_Y^{D',D})). \end{aligned}$$

For (7), we may assume as above that $X = Y$ is in \mathbf{Sm}_k , irreducible and that Id_Y is in $\mathcal{M}_*(Y)_{D+D'} \cap \mathcal{M}(Y)_{D,D'} \cap \mathcal{M}(Y)_{D',D}$. It suffices to show that $i_*(D(\text{Id}_Y)) = i'_*(D'(\text{Id}_Y))$. Our assumptions imply that Id_Y is in $\mathcal{M}(Y)_D \cap \mathcal{M}(Y)_{D'}$.

Suppose $Y = |D|$. Then $i = \text{Id}_Y$ and $D(\text{Id}_Y) = \tilde{c}_1(O_Y(D))(\text{Id}_Y)$. Thus, if $|D'| = Y$, we have $i_*(D(\text{Id}_Y)) = i'_*(D'(\text{Id}_Y))$. If $|D'| \neq Y$, then D' is a simple normal crossing divisor on Y and $D'(\text{Id}_Y) = [D' \rightarrow |D'|]$. Thus D' is in good position with respect to the empty sequence \emptyset and by lemma 1.14,

$$i'_*(D'(\text{Id}_Y)) = \tilde{c}_1(O_Y(D'))(\text{Id}_Y) = \tilde{c}_1(O_Y(D))(\text{Id}_Y) = i_*(D(\text{Id}_Y)).$$

If both $|D| \neq Y$ and $|D'| \neq Y$, then the same computation as above gives the desired result. \square

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